THIN POSITION WITH RESPECT TO A HEEGAARD SURFACE.

DAVID BACHMAN

ABSTRACT. We present a definition of *thin position* for a knot in a 3-manifold with respect to a Heegaard surface, motivated by Scharlamenn and Thompson's definition of thin position for 3-manifolds [ST94], and Gabai's definition of thin position for knots in S^3 [Gab87]. We then show that if a knot, K, in a 3-manifold, M, whose exterior contains no meridional, planar, essential surfaces, is put in thin position with respect to a strongly irreducible Heegaard surface then all thin levels of K are essential in the complement of K. A Corollary of this is that if a small knot, K, is put in thin position with respect to a strongly irreducible Heegaard surface, F, then either K is in bridge position with respect to F, or it is a core of one of the handlebodies bounded by F. This generalizes a Theorem of Thompson for knots in S^3 [Tho97].

Keywords: thin position, bridge position, Heegaard splitting, tunnel number

1. INTRODUCTION

In 1954, H. Schubert introduced a complexity for the position of a knot in S^3, called its *bridge number* [Sch54]. One way to define this is to put the knot into *bridge position, i.e.* a position in which all of its maxima appear above all of its minima, and count the number of maxima. 33 years after Schubert's definition, D. Gabai introduced another notion of complexity, called *width* [Gab87]. A knot is said to be in *minimal bridge position* if it has been isotoped to have minimal bridge number, and in *thin position* if it has been isotoped to have minimal width. In 1997, A. Thompson established a relationship between bridge number and width by proving that if a knot in S^3, whose exterior contains no essential, meridional, planar surfaces, is put in thin position then it must be in minimal bridge position [Tho97].

If a knot, K, is not in bridge position then there is a level 2-sphere which is just above some maxima of K, and just below some minima. Such a level 2-sphere is called a *thin level*. It is tempting to try to prove Thompson's theorem by showing that the thin levels of any knot in thin position in S^3 are essential in the knot complement. Although there is currently no known counter-example to this statement, this seems to be quite difficult to prove. Thompson's approach is to show that no thin level can compress completely to a set of boundary parallel annuli in the knot complement. Hence, the result of a maximal set of compressions of a thin level is an essential,

meridional, planar surface. If we assume that no such surface exists, then there must have not been any thin levels to begin with.

Before generalizing Thompson's result to manifolds other than S^3, we must understand what it means for a knot to be in bridge position or thin position in an arbitrary 3-manifold. A definition of bridge position is given by Morimoto and Sakuma in [MS91]. The idea is to put a height function on the 3-manifold in which a Heegaard surface, F, appears as a level surface. If F also separates the maxima of a knot, K, from its minima, then we say that K is in *bridge position with respect to F*.

Defining thin position for a knot in a 3-manifold is more difficult. Various definitions are presented, for example, in [Fei98] and [HS01]. In S^3, Gabai's definition of the width of a knot is a simple, integer valued complexity. It seems difficult to find a similar such complexity for knots in arbitrary 3-manifolds which has the desired properties. Instead, we use a definition which is closer in spirit to Scharlemann and Thompson's definition of the width of a height function on a 3-manifold. This complexity is a whole set of integers, where two such sets are compared lexicographically. Our definition of width, given in Section 5, will be similar, and will be dependent on both a height function and the position of a knot. If the height function is chosen so that a Heegaard surface, F, appears as the unique level surface of maximal genus, then this gives us a natural definition of what it means for a knot to be in *thin position with respect to F*.

Section 7 contains the main results of this paper. These address the earlier question regarding the essentiality of thin levels in a knot exterior. As we have mentioned, not much is known in general about this question in S^3, unless the knots under consideration are assumed to be *mp-small*, (*i.e.* no essential, planar, meridional surfaces in the knot complement), in which case Thompson's Theorem tells us that there are no thin levels at all. In other 3-manifolds, if we make precisely the same restriction on the knots under consideration, we get a weaker, but perhaps more enlightening statement:

Theorem 7.4 *Let F be a strongly irreducible Heegaard surface for a closed, irreducible 3-manifold, M, and $K \subset M$ be an mp-small knot or link, which is in thin position with respect to F. Then each thin level for K is incompressible and ∂-incompressible in the complement of K.*

A Corollary of this is our generalization of Thompson's Theorem to arbitrary 3-manifolds.

Corollary 7.5 *Let $W \cup_F W'$ be a strongly irreducible Heegaard splitting of an irreducible 3-manifold, M. If a small knot, K, is in thin position with respect to F, then either K is in bridge position with respect to F, or it is a core of W or W'.*

Results similar to Corollary 7.5 were independently obtained by C. Feist [Fei98] and C. Hayashi and K. Shimokawa [HS01]. Feist presents an integer-valued definition of width, leading to a definition of thin position for a knot with respect to a Heegaard

surface, F. To define this, he makes the restriction that the knots under consideration must lie within a product neighborhood of the surface (which, he notes, can always be arranged). He then shows that for small knots, thin position with respect to F is bridge position. This is done by showing that if a thin level exists, it cannot compress completely to a boundary-parallel annulus in the knot complement, and hence, it compresses to some essential surface with meridional boundary.

Hayashi and Shimokawa, on the other hand, use a definition of width which is closer in spirit to the one presented here for knots with respect to a height function. (It should be noted that the complexity they use does not exactly accomplish all that they claim, but it is not hard to modify it so that it works.) They show that the thin levels of a thin presentation of a knot are indeed essential, meridional surfaces in the knot complement. However, in the process of "thinning" a knot they lose control of the underlying height function. Consequently, they are not able to present a definition of thin position for a knot with respect to a fixed Heegaard surface.

Another statement that can be deduced from Theorem 7.4 is the following Corollary, which establishes a relationship between width, bridge number, and *tunnel number* (*i.e.* the Heegaard genus of the complement of a knot).

Corollary 7.6 *Let F be a strongly irreducible Heegaard surface in an irreducible 3-manifold, M. Let K be a small knot in M, such that the genus of F is less than the tunnel number of K. Then thin position of K with respect to F is bridge position with respect to F.*

2. Background material.

In this section, we give some of the standard definitions that will be used throughout the paper. The expert in 3-manifold theory can easily skip this.

A 2-sphere in a 3-manifold which does not bound a 3-ball on either side is called *essential*. If a manifold does not contain an essential 2-sphere, then it is referred to as *irreducible*.

A loop on a surface is called *essential* if it does not bound a disk in the surface. Given a surface, F, in a 3-manifold, M, a *compressing disk* for F is a disk, $D \subset M$, such that $F \cap D = \partial D$, and such that ∂D is essential on F. If we let $D \times I$ denote a thickening of D in M, then to *compress F along D* is to remove $(\partial D) \times I$ from F, and replace it with $D \times \partial I$.

A *compression body* is a 3-manifold which can be obtained by starting with some surface, F (not necessarily connected), forming the product, $F \times I$, attaching some number of 2-handles to $F \times \{1\}$, and capping off all resulting 2-sphere boundary components with 3-balls. The boundary component, $F \times \{0\}$, is often referred to as ∂_+. The other boundary component is referred to as ∂_-. If $\partial_- = \emptyset$, then we say the compression body is a *handlebody*. A compression body is *non-trivial* if it is not a product.

A surface, F, in a 3-manifold, M, is a *Heegaard surface for M*, if F separates M into two compression bodies, W, and W', such that $F = \partial_+ W = \partial_+ W'$. Such a splitting is *non-trivial* if both W and W' are non-trivial.

Definition 2.1. A separating surface, F, in a 3-manifold, M, is *strongly irreducible* if every compressing disk for F on one side intersects every compressing disk for F on the other.

Lemma 2.2. *If M contains a non-trivial strongly irreducible Heegaard surface, then ∂M is incompressible in M.*

This result is typically proved by appealing to a Lemma of Haken ([Hak68], see also [Jac77]), but notice that it also follows from the following well-known result:

Theorem 2.3. *Let M be a 3-manifold which contains a non-trivial strongly irreducible Heegaard surface, F, and an essential surface, S. Then F can be isotoped so that all curves of $F \cap S$ are essential on both surfaces.*

Lemma 2.2 follows from this Theorem by noting that a compressing disk for the boundary of M is an essential surface, but no compressing disk for the boundary of a manifold can be made disjoint from a Heegaard surface. In Section 6 we will present a generalization of Lemma 2.2, which will be proved by appealing to a result which is analogous to Theorem 2.3.

3. Thin Position for 3-manifolds

In this section we review the results of Scharlemann and Thompson from [ST94]. Our approach is slightly different, in the sense that the complexities we use are different than those of the aforementioned paper. In the end, however, we will show that the results are the same.

Let $h : M \to I$ be a Morse function on a 3-manifold, M. Let $\{t_i'\}$ denote the critical values of h, and for each i, choose some $t_i \in (t_i', t_{i+1}')$. For all $t \in I$, let $F_t = h^{-1}(t)$.

Definition 3.1. For each surface $F \subset M$, let $c(F) = \sum_j (2 - \chi(F^j))^2$, where $\{F^j\}$ is the set of components of F.

Note that this complexity is defined so that if $F_{t_{i+1}}$ is obtained from F_{t_i} by a compression then $c(F_{t_{i+1}}) < c(F_{t_i})$, and so that if F_{t_i} is a union of 2-spheres then $c(F_{t_i}) = 0$.

Definition 3.2. If i is a number such that $c(F_{t_i}) > c(F_{t_{i-1}})$ and $c(F_{t_i}) > c(F_{t_{i+1}})$, then we say the surface F_{t_i} is a *thick level* of h. Similarly, if $c(F_{t_i}) < c(F_{t_{i-1}})$ and $c(F_{t_i}) < c(F_{t_{i+1}})$, then we say the surface F_{t_i} is a *thin level* of h.

Definition 3.3. The *width* of h, $w(h)$, is defined to be the set of integers $\{c(F_{t_i})|F_{t_i}$ is a thick level of $h\}$, where repetitions are included, and the set is ordered in non-increasing order.

The concept of width gives us a way of comparing two Morse functions on the same 3-manifold. We will say that $h_1 < h_2$ if $w(h_1) < w(h_2)$, where the latter comparison is made lexicographically.

Lemma 3.4. *Every 3-manifold admits a Morse function of minimal width.*

Proof. This follows immediately from the fact that every decreasing sequence of sets of integers, ordered lexicographically, must terminate. $\square$

Definition 3.5. A *thin decomposition* of a 3-manifold, M, is a sequence of pairwise disjoint surfaces, $\{F_i\}$, where the sets $\{F_{2i+1}\}$ and $\{F_{2i}\}$ are the thick and thin levels, respectively, of a Morse function, h, such that $w(h)$ is minimal.

Note that if h is any Morse function on M, L is a thin level of h, and H is the next (or previous) thick level, then the submanifold of M cobounded by L and H is a compression body, W, such that $\partial_- W = L$, and $\partial_+ W = H$. Hence, h determines a *Heegaard-Scharlemann-Thompson (HST) splitting*, *i.e.* a sequence of pairwise-disjoint surfaces, $\{F_i\}$, where for each odd i, F_i is a non-trivial Heegaard splitting of the submanifold cobounded by F_{i-1} and F_{i+1} (Note that we will always begin our indexing with $i = 0$, and assume that F_0 is the first thin level of h. Using this convention, $F_0 = \emptyset$ for closed 3-manifolds). Conversely, for any HST splitting, $\{F_i\}$, there is a height function, h, such that the sets $\{F_{2i+1}\}$ and $\{F_{2i}\}$ are the thick and thin levels of h. Since the width of h only depends on its set of thick levels, this observation lets us work with HST splittings, rather than Morse functions. Henceforth, we may abuse terminology slightly by talking about the *width of an HST splitting*, rather than the width of a Morse function for which the HST splitting consists of its thick and thin levels. A thin decomposition of a 3-manifold is then an HST splitting of minimal width.

The next Theorem is due to Scharlamenn and Thompson. We include a brief proof only because our definition of width is slightly different than theirs.

Theorem 3.6. (Scharlemann-Thompson [ST94]) *Let $\{F_i\}$ be a thin decomposition of a 3-manifold, M. Then for each odd i, F_i is a non-trivial strongly irreducible Heegaard splitting of the submanifold of M cobounded by F_{i-1} and F_{i+1}.*

Proof. Suppose this is not true. Then for some odd number, p, there exists disks, D and E, such that:

(1) D and E are compressing disks for F_p,
(2) D lies in the submanifold of M cobounded by F_{p-1} and F_p,
(3) E lies in the submanifold of M cobounded by F_p and F_{p+1}, and
(4) $D \cap E = \emptyset$.

Let F_D be the surface obtained from F_p by compression along D, F_E the surface obtained from F_p by compression along E, and F_{DE} the surface obtained from F_p by compression along both D and E.

We now alter the sequence, $\{F_i\}$. There are four cases:

(1) $F_D \neq F_{p-1}$, $F_E \neq F_{p+1}$.
  Remove F_p from $\{F_i\}$. In its place, insert $\{F_D, F_{DE}, F_E\}$ and reindex.
(2) $F_D = F_{p-1}$, $F_E \neq F_{p+1}$.
  Replace $\{F_{p-1}, F_p\}$ with $\{F_{DE}, F_E\}$.
(3) $F_D \neq F_{p-1}$, $F_E = F_{p+1}$.
  Replace $\{F_p, F_{i+1}\}$ with $\{F_D, F_{DE}\}$.
(4) $F_D = F_{p-1}$, $F_E = F_{p+1}$.
  Replace $\{F_{p-1}, F_p, F_{p+1}\}$ with F_{DE} and reindex.

In all cases, it is a routine matter to check that we have defined a new HST splitting, $\mathcal{G}$. The set $w(\mathcal{G})$ is obtained from the set $w(\{F_i\})$ by removing some element, and replacing it with at most two smaller numbers. Hence, under the lexicographical ordering, $w(\mathcal{G}) < w(\{F_i\})$. $\qquad\square$

Scharlemann and Thompson then show that Theorem 3.6 and Lemma 2.2 imply:

Theorem 3.7. (Scharlemann-Thompson [ST94]) *Let $\{F_i\}$ be a thin decomposition of a 3-manifold, M. Then for each even i, F_i is incompressible in M.*

Corollary 3.8. *A thin decomposition of a non-Haken 3-manifold is a strongly irreducible Heegaard surface.*

An alternate way to state this Corollary would be to say that if h is a Morse function on a non-Haken 3-manifold, M, such that $w(h)$ is minimal, then h has precisely one thick level. This is completely analogous to Thompson's theorem [Tho97], which says that if K is a small knot such that $w(K)$ is minimal, then K has precisely one thick level. (In the next section we will define the terms *thick level* and *width* for knots in S^3.)

Still another way to view Corollary 3.8 is to say that a Heegaard surface for a non-Haken 3-manifold which is *not* strongly irreducible cannot define a generalized Heegaard splitting of minimal width. From this point of view one can see that Corollary 3.8 is equivalent to a result of Casson and Gordon [CG87], which says that a Heegaard splitting of a non-Haken 3-manifold which is not strongly irreducible cannot be a minimal genus splitting.

4. Thin Position for Knots in S^3

Suppose that $K \subset S^3$ is an arbitrary knot or link, and $h : S^3 \to I$ is the standard height function (so that for all $t \in (0, 1)$, $h^{-1}(t)$ is a 2-sphere). Isotope K so that h is a Morse function when restricted to K. Let $\{t_i'\}$ denote the critical values of h restricted to K, and let t_i be some point in the interval (t_i', t_{i+1}'). Finally, let $F_t = h^{-1}(t)$.

Definition 4.1. If $K \subset M$ is some properly embedded 1-manifold, then let M_K denote M with a regular neighborhood of K removed. If X is any subset of M, then let $X_K = M_K \cap X$.

Note that since each level, F_t, is a sphere, $|K \cap F_t| = 2 - \chi((F_t)_K)$. This motivates us to define the following complexity:

Definition 4.2. For all t, let $c^K(F_t) = 2 - \chi((F_t)_K)$.

The definitions of *thick level*, *thin level*, and *width* that we will use are now exactly the same as those given in Section 3, where one substitutes the function c^K for each occurrence of the function c. Since the same height function will be used for all knots in S^3, we will consider the function, w, to be a complexity for positions of knots, rather than for Morse functions, and write $w(K)$, rather than $w(h)$.

As in the previous section, our definition of width is an ordered set of integers. Gabai's original definition (see [Gab87]) of $w(K)$ is the quantity $\frac{1}{2} \sum_i c^K(F_{t_i})$. Our definition is equivalent, in the sense that it can be used to rule out certain configurations for knots with minimal width. In a moment we will make this more precise.

Definition 4.3. A knot, K, is said to be in *thin position* if it have been isotoped so that $w(K)$ is minimal.

Definition 4.4. A *high disk* for a level, F_t is a disk, D, such that $\partial D = \alpha \cup \beta$, where $\alpha \subset K$, $\beta \subset F_t$, and a collar of β in D lies above F_t. *Low disks* are defined similarly.

The salient feature of both our definition of thin position for knots in S^3 and Gabai's is that if K is in thin position, then one never sees disjoint high and low disks for K, or high and low disks that meet in a point of K (otherwise, we could perform one of the moves depicted in Figure 1 to obtain a position of K with smaller width).

5. Thin Position for knots in 3-manifolds

We now synthesize the definitions given in the previous two sections. Suppose that $K \subset M$ is an arbitrary knot or link, and $h : M \to I$ is a Morse function on a 3-manifold, M. Isotope K so that h is a Morse function when restricted to K, and so that no critical value of h coincides with a critical value of $h|_K$. Let $\{t_i'\}$ denote the union of the critical values of h, and the critical values of $h|_K$. Let t_i be some point in the interval (t_i', t_{i+1}'). For all t, let $F_t = h^{-1}(t)$.

Definition 5.1. For any surface F in M, let $C^K(F) = \sum_j (2 - \chi((F_j)_K))^2$, where the sum is taken over all components, $\{F_j\}$, of F.

We now define *thick levels*, *thin levels*, and *width* exactly as in Section 3, using the function, C^K, instead of c. Width now depends on both the position of a knot, K, and on a choice of Morse function, h, so it will be written $w(K; h)$.

Definition 5.2. If K has been isotoped so as to minimize $w(K; h)$, then we say K is in *thin position with respect to h*.

In this paper we will mainly be concerned with height functions, h, *associated* to a Heegaard surface, F, in the sense that F is the only thick level of h.

Definition 5.3. Let F be a Heegaard surface for a 3-manifold, M. If we choose a height function, h, associated to F, and a position of a knot, K, so that $w(K; h)$ is minimal, then we say that K is in *thin position with respect to F*.

As in S^3, we also make the following definition:

Definition 5.4. If h is a height function associated to a Heegaard surface, F, then we say a knot, K, is in *bridge position with respect to F* if F is the unique thick level of K with respect to h, *i.e.* if all of the maxima of K are above F, and all of the minima of K are below.

It is the goal of this paper to understand when thin position for K with respect to a strongly irreducible Heegaard surface conincides with bridge position.

6. Heegaard splittings of a pair, (M, K)

To proceed, we must understand arcs in a compression body. If W is a compression body, recall that W can be built by starting with a product, $F \times I$, and attaching 2- and 3-handles to $F \times \{1\}$. Anything that remains of $F \times \{1\}$ after the attachment becomes part of $\partial_- W$. We say an arc, k, is *straight* in W if $k = \{p\} \times I$, where $p \in F$ is a point such that $\{p\} \times \{1\} \in \partial_- W$.

We are now ready to generalize the definition of a compression body:

Definition 6.1. A *K-compression body*, $(W; K)$, is a compression body, W (possibly a product), and a 1-manifold, $(K, \partial K) \subset (W, \partial W)$, such that

(1) K is a disjoint union of embedded arcs
(2) each arc of K has at least one endpoint on $\partial_+ W$
(3) if k is an arc of K with $\partial k \subset \partial_+ W$, then there is a disk, $D \subset W$, with $\partial D = k \cup \alpha$, $D \cap K = k$, and $D \cap \partial_+ W = \alpha$.
(4) if k is an arc of K with one endpoint on $\partial_+ W$, then k is straight.

A K-compression body, $(W; K)$, is *non-trivial* if either W is not a product, or at least one arc of K is not straight.

Definition 6.2. If K is a 1-manifold properly embedded in a 3-manifold, M, then a *Heegaard splitting of the pair*, (M, K), is an expression of M as a union of K_i-compression bodies, $(W_1; K_1)$ and $(W_2; K_2)$, such that $\partial_+ W_1 = \partial_+ W_2$, and $K = K_1 \cup K_2$. Such a splitting is *non-trivial* if both $(W_1; K_1)$ and $(W_2; K_2)$ are non-trivial.

When the context is clear, we will refer to the surface, $\partial_+ W$, as a *Heegaard surface* of $(M; K)$.

Definition 6.3. Suppose F is a surface in a 3-manifold M, and K is some properly embedded 1-manifold in M, transverse to F. A *relative compressing disk* for F is a

disk, D, such that $\partial D = \alpha \cup \beta$, where $D \cap K = \alpha$, and $D \cap F = \beta$. If there are no relative compressions for some surface, then we say that it is *relatively incompressible*.

Definition 6.4. Suppose F is a separating surface in a 3-manifold, M, and K is some properly embedded 1-manifold in M, transverse to F. Then F is *weakly reducible with respect to K* if there exists disks, D and E, on opposite sides of F, such that either

(1) D and E are disjoint compressing disks for F_K
(2) D and E are relative compressing disks for F that are either disjoint, or meet in a point of K
(3) D is a relative compression for F, and E is a disjoint compressing disk for F_K in M_K
(4) D is a compressing disk for F_K in M_K, and E is a disjoint relative compressing disk for F.

We say F is *strongly irreducible with respect to K* if it is not weakly reducible.

The following is a generalization of Theorem 2.3:

Lemma 6.5. *Suppose F is a Heegaard surface of $(M; K)$, which is strongly irreducible with respect to K, and $(S, \partial S) \subset (M, \partial M \cup K)$ is an essential surface. Then F can be isotoped rel K so that no arc of $S \cap F$ cobounds a relative compressing disk for F, and so that no loop of $S \cap F$ bounds a compressing disk for F_K in M_K.*

Proof. The proof is similar to many of the standard arguments in 3-manifold topology. It is a standard construction to define a height function, $h : M \to I$, so that $h^{-1}(\frac{1}{2}) = F$, and so that for every $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, there is a compression body, $W_t \subset M$, such that $\partial_+ W_t = F$, and $\partial_- W_t = h^{-1}(t)$.

Let $t_1 = \inf_{t < 1/2} \{t | (W_t; T_1 \cap W_t)$ is a trivial relative compression body$\}$, and let $t_2 = \sup_{t > 1/2} \{t | (W_t; T_1 \cap W_t)$ is a trivial relative compression body$\}$. Let $\{s_i\}$ denote the points of the interval $[t_1, t_2]$ where $h^{-1}(t)$ is not transverse to S. If for some point, t, of the subinterval, $[s_i, s_{i+1}]$, there is an arc of $S \cap h^{-1}(t)$ which cobounds a relative compressing disk for $h^{-1}(t)$, or a loop of $S \cap h^{-1}(t)$ which bounds a compressing disk for $h^{-1}(t)_K$ in M_K, on the same side of $h^{-1}(t)$ as $h^{-1}(0)$, then label this subinterval "A". Similarly, if there is a compressing or relative compressing disk for $h^{-1}(t)$, bounded or cobounded by a component of $S \cap h^{-1}(t)$, on the same side of $h^{-1}(t)$ as $h^{-1}(1)$, then label the subinterval "B". As F is strongly irreducible with respect to K, and $h^{-1}(t)$ is parallel to F for all $t \in (t_1, t_2)$, we know that $h^{-1}(t)$ is strongly irreducible with respect to K in this range. Hence, there does not exist a value of i such that the subinterval $[s_i, s_{i+1}]$ has both of the labels "A" and "B", since compressions and relative compressions, whose boundaries are contained in S, on opposite sides of $h^{-1}(t)$, form a weak reduction pair. Similarly, using an argument identical to that of Lemma 4.4 of Gabai's paper, [Gab87], one can show that adjacent

subintervals cannot have different labels. Now, to conclude that there is an unlabelled subinterval, we must show that there is at least one subinterval with each label.

There are several cases to consider, depending on exactly what happens at t_1 and t_2. Suppose first that there is a minimum of K at t_1. Then for values of t just after t_1, we see a relative compressing disk for $h^{-1}(t)$, contained in S, on the same side of $h^{-1}(t)$ as $h^{-1}(0)$. Hence, the first subinterval of $[t_1, t_2]$ would have the label "A".

If, instead, we see $h^{-1}(t)_K$ compress as t decreases from $\frac{1}{2}$ to t_1, then for values of t just after t_1 there is a compressing disk, D, for $h^{-1}(t)_K$ which is disjoint from S, and on the same side of $h^{-1}(t)$ as $h^{-1}(0)$. If, for t just after t_1, $h^{-1}(t)$ does not intersect S as stated in the theorem, then there are two cases:

(1) There is an arc of $S \cap h^{-1}(t)$ which cobounds a relative compressing disk, D', for F. Then $\partial D' \cap \partial D = \emptyset$, and so by strong irreducibility, D' must be on the same side of $h^{-1}(t)$ as D. Hence, again the first subinterval of $[t_1, t_2]$ would have the label "A".

(2) There is a loop of $S \cap h^{-1}(t)$ which bounds a compressing disk, D'', for F_K. As in Case 1, $\partial D'' \cap \partial D = \emptyset$, so we may conclude that D'' is on the same side of $h^{-1}(t)$ as D. Once again, the first subinterval of $[t_1, t_2]$ would have the label "A".

A symmetric argument shows that the last subinterval of $[t_1, t_2]$ must have the label "B". By the remarks which precede the last paragraph, we may now conclude that there is an unlabelled subinterval. For values of t in this subinterval, there is no compressing or relative compressing disk bounded or cobounded by any component of $S \cap h^{-1}(t)$. $\qquad \square$

We are now prepared to present our generalization of Lemma 2.2.

Lemma 6.6. *If F is a Heegaard surface of $(M; K)$, which is strongly irreducible with respect to K, then $(\partial M)_K$ is incompressible in M_K, and ∂M is relatively incompressible in M.*

Proof. Suppose that D is a compressing disk for $(\partial M)_K$ in M_K, or a relative compressing disk for ∂M. Then we may apply Lemma 6.5 to isotope F so no loop of $D \cap F$ bounds a compressing disk for F_K. Hence, if γ is a loop of $D \cap F$ which is innermost on D, then γ must bound a disk, E, on F_K. Let α be a loop of $D \cap F$ which is innermost on E. Then α bounds a subdisk of E which we can use to surger D. This eliminates one component of $D \cap F$. We may now repeat the process, eventually removing all loops of $D \cap F$.

If D were a compressing disk, then we would have now made D and F disjoint. If, instead, D were a relative compressing disk, then there may still be arcs of intersection remaining. Our previous application of Lemma 6.5 had put us in a position such that no arc of $D \cap F$ cobounds a relative compressing disk for F. Let β be an arc of $D \cap F$ which is outermost on D. Then β cobounds a relative compressing disk for F. As this is a contradiction, we conclude that there are now no arcs of $D \cap F$. In any

case, D has become a compressing or a relative compressing disk for the negative boundary component of a relative compression body, which is impossible. □

7. When thin is bridge

Definition 7.1. [Rieck-Sedgwick] A 1-manifold $(K, \partial K) \subset (M, \partial M)$ is *mp-small* if there are no planar, incompressible, non-boundary parallel, meridional surfaces in M_K.

Theorem 7.2. *Let F be a strongly irreducible Heegaard surface for a closed, irreducible 3-manifold, M, and $K \subset M$ be an mp-small knot or link, which is in thin position with respect to F. Then each thick level for K is a strongly irreducible Heegaard surface for the submanifold of $(M; K)$ cobounded by consecutive thin levels.*

Proof. If the Theorem is false, then there are disks, D and E, for some thick level, G, of K, as in Definition 6.4, which lie in a submanifold of M cobounded by consecutive thin levels. We will say a compressing disk for G_K in M_K is *true* if it is also a compressing disk for G. Otherwise, it is *false*.

We now check all cases (up to symmetry):

Case 1. *D and E are relative compressing disks.* In this case, we can do one of the moves depicted in Figure 1, thereby producing a position of K whose width is smaller.

Case 2. *D and E are true compressing disks.* Then G must be parallel to F, and hence, D and E are isotopic to disjoint compressing disks for F. As F was assumed to be strongly irreducible, this is a contradiction.

Case 3. *D and E are false compressing disks.* Let D' denote the subdisk of G bounded by ∂D, and E' the subdisk of G bounded by ∂E. There are now two subcases (up to symmetry):

Subcase 3.1. $D' \cap E' = \emptyset$. Note that $D \cup D'$ is a sphere which bounds a ball, B. Inside B there must be a relative compressing disk, H, since ∂D is essential on G_K. Similarly, inside the sphere $E \cup E'$, there must be a relative compressing disk, L. As $D' \cap E' = \emptyset$, it must be that $H \cap L = \emptyset$, so we are reduced to Case 1 above.

Subcase 3.2. $E' \subset D'$. This is by far the most difficult case. To obtain a contradiction, we begin by mimicking Thompson's argument from [Tho97]. First, we assume that D and E are chosen to be outermost, in the sense that there does not exist a compressing disk, E'' (say), for G_K, on the same side of G as E, such that $\partial E''$ bounds a disk on G which contains E'. Let A denote the annulus which is the closure of $D' - E'$. Let S be the sphere, $D' \cup A \cup E$.

Compress S as much as possible in M_K, to obtain a collection of spheres, $\tilde{P}$, in M. Let P be an innermost component of $\tilde{P}$, in the sense that the ball bounded by P in M does not contain any other component of $\tilde{P}$. Since K is mp-small, and P_K
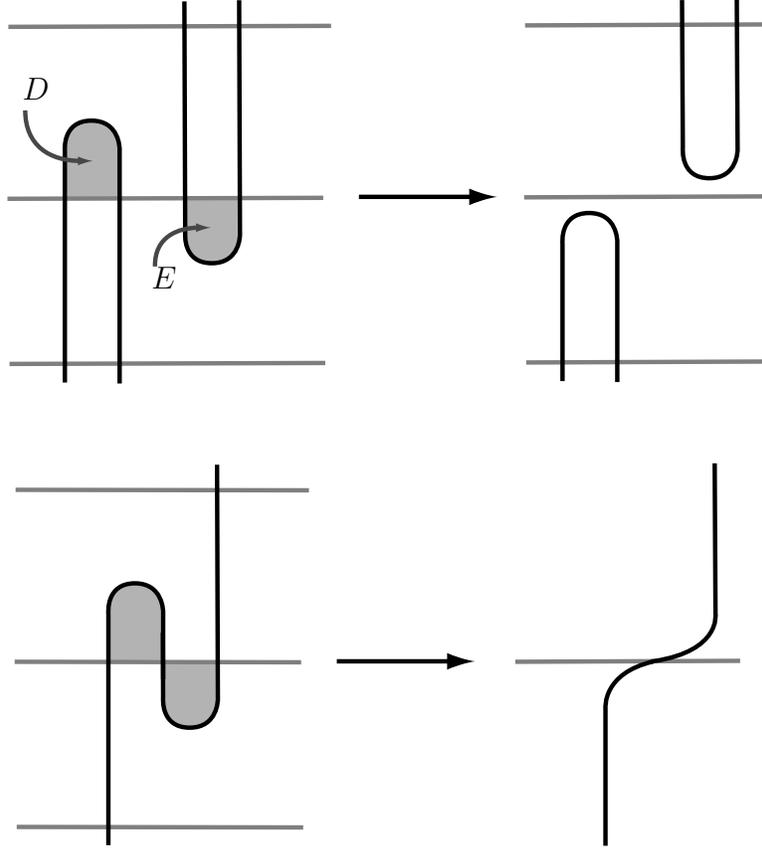
FIGURE 1. Moves which produce lower width.

is incompressible in M_K, it must be that P is relatively compressible in M. Let R then denote a relative compressing disk for P.

We now reverse the compressions used to obtain $\tilde{P}$ from S. Each time a compression is reversed, we attach a tube to some components of $\tilde{P}$. These tubes may intersect R, but only in its interior. In the end, R persists as a disk in M such that $\partial R = \alpha \cup \beta$, where $K \cap R = \alpha$, $\beta \subset S$, and a collar of β in R is disjoint from S.

As β is an arc on S, and S is made up of two disks, D and E, which are disjoint from K, and an annulus, A, we may isotope R so that $\beta \subset A$. We now claim that α meets at least one thin level of K. If not then there are two cases to consider, depending on which side of G contains a collar of β in R. Let W_D be the compression body cobounded by G and the previous thin level, which contains D. Let W_E denote the compression body cobounded by G an the next thin level, containing E.

Subcase 3.2.1. *A collar of β in R is contained in W_D.* As in subcase 3.1, there is a relative compression, L, inside the sphere, $E \cup E'$. But $L \cap G \subset E'$, whose

interior is disjoint from A. As A contains β, we conclude that $R \cap L = \emptyset$, and are basically reduced to case 1. (R may not actually be a relative compressing disk, since the interior of R may meet G. However, we can still use R to guide one of the width-lowering isotopies of K depicted in figure 1.)

Subcase 3.2.2. *A collar of β in R is contained in W_E.* We first claim that $\alpha \cap G$ must be empty. Otherwise there would be an arc, β', of $R \cap G$, and a subarc, $\alpha' \subset \alpha$, which cobounded a subdisk, R', of R, such that a collar of β' in R' lies in W_D. Furthermore, there exists such a β' which is disjoint from E', so we are reduced to the previous case, where there is a relative compressing disk, L, contained in the ball bounded by $E \cup E'$, which can be paired with R' to form a width-reducing move.

We conclude then that α lies entirely in W_E. Let L' be a disk in W_E such that $\partial L' = \alpha \cup \beta$, where $L' \cap K = \alpha$ and $L' \cap G = \beta \subset A$. (If it were not for the fact that the interior of R may not lie entirely in W_E, we could take $L' = R$.) Let δ be any arc in G_K joining the interior of β to ∂E. Then a frontier of a neighborhood of $L' \cup \delta \cup E$ in W_E contains a disk which is "more outermost" than E, contradicting our assumption.

We conclude then that α meets at least one thin level of K. Let β'' be an arc of intersection of R with the thin levels of K, which is outermost on R. $\partial \beta''$ bounds a subarc, $\alpha'' \subset \alpha$. We claim that α'' must meet some thick level. If not, then α'' lies in some relative compression body, W (cobounded by a thick and a thin level of K). But α'' is then a subarc of K in W, both of whose endpoints are on $\partial_- W$, a contradiction.

Let S denote the thin level of K that contains β'', and let T denote the thick level whose intersection with α'' is non-trivial. Let R'' be the subdisk of R bounded by $\beta'' \cup \alpha''$. Use R'' to guide an isotopy of K, reducing $|K \cap S|$ by 2, and $|K \cap T|$ by at least 2. It is now easy to check that such a move reduces width.

Case 4. *D is a relative compressing disk, and E is a false compressing disk.* The boundary of a neighborhood of D in M is a sphere, which is cut into two disks by G. One of these disks, D', is a false compressing disk for G, on the same side of G as D. Since $E \cap D = \emptyset$, we may conclude that $E \cap D' = \emptyset$, and hence we are reduced to Case 3 above.

Case 5. *D is a relative compressing disk, and E is a true compressing disk.* Our proof will be almost identical to that of Theorem 3.6. First, notice that the thick and thin levels of K form an *HST splitting* of $(M; K)$, *i.e.* a sequence of pairwise-disjoint surfaces, $\{F_i\}$, where for each odd i, F_i is a Heegaard splitting of the pair which consists of the submanifold cobounded by F_{i-1} and F_{i+1}, and the subset of K which lies in this submanifold. The sets, $\{F_{2i-1}\}$ and $\{F_{2i}\}$ are then the thick and thin levels of K. Our goal is to construct a new HST splitting, $\mathcal{S}$, of $(M; K)$, whose width is smaller than the width of $\{F_i\}$. (As before, the *width* of a HST splitting, $\{F_i\}$, is defined to be $w(K, h)$, where h is chosen so that the thick and thin levels of K with

respect to h are precisely the sets, $\{F_{2i-1}\}$ and $\{F_{2i}\}$). The one caveat is that we must be careful that if h' is a height function such that the thick and thin levels of K with respect to h' are $\mathcal{S}$, then h' itself must have only one thick level, namely F. This is necessary in order to contradict the assumption that K was in thin position *with respect to F* (rather than with respect to just any height function). This part of the proof is dependent on the observation that the thick levels of h' are a subset of the thick levels of K with respect to h'.

Let p be the odd number such that $G = F_p$. Let F_D be the surface obtained from F_p by relative compression along D (*i.e.* use D to guide an isotopy of F_p, removing two points of intersection with K), F_E the surface obtained from F_p by compression along E, and F_{DE} the surface obtained from F_p by relative compression along D and compression along E.

We now alter the HST splitting via one of the four operations defined in the proof of Theorem 3.6. The proof that this decreases width is precisely the same. We will denote the new HST splitting by $\mathcal{S}$. Let h be a height function such that the thick and thin levels of K with respect to h alternate in $\{F_i\}$, and let h' be a height function such that the thick and thin levels of K with respect to h' alternate in $\mathcal{S}$.

We now show that if we ignore K, h' has no more thick levels than h. The only thick levels of $\mathcal{S}$ that were not thick levels of $\{F_i\}$ are (at worst) F_D and F_E. But, when we ignore K, F_D is parallel to F_p, so the introduction of F_D as a thick level for $\mathcal{S}$ cannot have caused the introduction of a new thick level for h'. Also, again ignoring K, F_E can be obtained from F_D by a compression. Since F_D was not a new thick level for h', then the same must be true of F_E.

We must still show that F is a thick level of h'. As previously noted, the thick levels of h are a subset of the thick levels of $\{F_i\}$. Lets suppose that q is the odd integer such that $F = F_q$. If $q \neq p$ then F_q is a thick level of both $\{F_i\}$ and $\mathcal{S}$. If $q = p$, then F_D, which is parallel to F_p (and hence to $F = F_q$), is a thick level.

Case 6. *D is a false compressing disk, and E is a true compressing disk.* Let D' denote the subdisk of G bounded by ∂D. As $D \cap E = \emptyset$, it must be the case that $D' \cap E = \emptyset$. As described in subcase 3(a), inside the sphere, $D \cup D'$, there must be a relative compressing disk, H. As $D' \cap E = \emptyset$, we may conclude that $H \cap E = \emptyset$, and we are reduced to Case 5 above.

$\square$

The proof of the next Theorem is almost identical to that of Theorem 3.7.

Theorem 7.3. *Let F be a strongly irreducible Heegaard surface for a closed, irreducible 3-manifold, M, and $K \subset M$ be an mp-small knot or link, which is in thin position with respect to F. Then each thin level for K is incompressible and ∂-incompressible in the complement of K.*

Proof. Let $\{F_i\}$ denote the HST splitting which consists of the thick and thin levels of K. For each odd i, let M_i denote the submanifold of M cobounded by F_{i-1} and

F_{i+1}. Suppose c is a loop that bounds a compressing disk, or an arc which cobounds a relative compressing disk, for some surface, F_q, where q is even. If c is a loop, then let C denote a disk in M, such that $\partial C = c$. If c is an arc, then let C be a disk in M such that $\partial C = \gamma \cup c$, where $K \cap C = \gamma$. In either case, choose C so that $|C \cap (\bigcup_{\text{even } i} F_i)|$ is minimal.

It follows from Theorem 7.3 and Lemma 6.6 that for each odd i, ∂M_i is incompressible and relatively incompressible in M_i. Hence, C cannot lie entirely in M_{q-1} or M_{q+1}. We conclude then that there is some loop or arc of intersection of the interior of C with $\bigcup_{\text{even } i} F_i$. Let α denote an innermost such loop. Let C' denote the subdisk of C bounded by α. C' lies in M_p, for some odd number, p. As ∂M_p is incompressible in M_p, α must bound a disk, A, on ∂M_p.

Now, let β be an innermost loop of $C \cap A$, and let A' be the subdisk of A bounded by β. Then we can use A' to surger C, and thereby obtain a new disk, with the same boundary as C, contradicting our minimality assumption.

We conclude then that c cannot be a loop, and if c is an arc, then C contains no loops of intersection with $\bigcup_{\text{even } i} F_i$. Let δ then denote an arc of intersection which is outermost on C. δ and a subarc of α cobound a subdisk, C'', of C. C'' is then a relative compressing disk for ∂M_i, for some i, contradicting Lemma 6.6. $\qquad\square$

The following Corollary, which is the appropriate generalization of Thompson's Theorem [Tho97] to knots in 3-manifolds other than S^3, is an immediate consequence of Theorem 7.4.

Corollary 7.4. *Let $W \cup_F W'$ be a strongly irreducible Heegaard splitting of an irreducible 3-manifold, M. If a small knot, K, is in thin position with respect to F, then either K is in bridge position with respect to F, or it is a core of W or W'.*

Proof. It follows from [CGLS87] that if K is small, then it is also mp-small. Hence, Theorem 7.4 implies that the thin levels of K with respect to F are essential in the complement of K. But again, it then follows from [CGLS87] that since K is small, the only possibility for a thin level is a boundary parallel torus, T, in M_K. If such a torus is a level of a height function associated to F, then there must be a compression body between F and T. This is equivalent to saying that K is a core of one of the handlebodies bounded by F.

If K is not isotopic to a core of W or W', then we conclude that K has no thin levels. Hence, there must be a unique thick level, namely, F. This is the precise definition of what it means for K to be in bridge position with respect to F. $\qquad\square$

Corollary 7.5. *Let F be a strongly irreducible Heegaard surface in an irreducible 3-manifold, M. Let K be a small knot in M, such that the genus of F is less than the tunnel number of K. Then thin position of K with respect to F is bridge position with respect to F.*

Proof. Corollary 7.5 says that if thin position for a small knot, K, with respect to F is not bridge position, then K is isotopic to a core of one of the handlebodies bounded by F. But then F is a Heegaard splitting of M_K, so the genus of F is at least the tunnel number of K. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

REFERENCES

[CG87]   A. J. Casson and C. McA. Gordon. Reducing Heegaard splittings. *Topology and its Applications*, 27:275–283, 1987.

[CGLS87] M. Culler, C. McA. Gordon, J. Luecke, and P.B. Shalen. Dehn surgery on knots. *Ann. of Math.*, 125:237–300, 1987.

[Fei98]   C. Feist. *Results on thin position*. PhD thesis, U.C. Davis, 1998.

[Gab87]   D Gabai. Foliations and the topology of three-manifolds iii. *J. Diff. Geom.*, 26:479–536, 1987.

[Hak68]   W. Haken. *Some Results on Surfaces in 3-Manifolds*. M.A.A., Prentice Hall, 1968.

[HS01]   C. Hayashi and K. Shimokawa. Thin position of a pair (3-manifold, 1-submanifold). *Pacific J. Math.*, 197:301–324, 2001.

[Jac77]   W. Jaco. *Lectures on Three Manifold Topology*, volume 43. CBMS Regional Conference Series in Mathematics, 1977.

[MS91]   K. Morimoto and M. Sakuma. On unknotting tunnels for knots. *Math. Ann.*, 289:143–167, 1991.

[Sch54]   H. Schubert. Ueber eine numerische Knoteninvariante. *Math. Z.*, 61:245–288, 1954.

[ST94]   M. Scharlemann and A. Thompson. Thin position for 3-manifolds. *A.M.S. Contemporary Math.*, 164:231–238, 1994.

[Tho97]   A. Thompson. Thin position and bridge number for knots in the 3-sphere. *Topology*, 36:505–507, 1997.

MATHEMATICS DEPARTMENT, UNIVERSITY OF ILLINOIS AT CHICAGO
E-mail address: bachman@math.uic.edu