THEORY BUILDING IN THE MATHEMATICS CURRICULUM?

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Abstract
Mathematicians distinguish two modes of their practice – problem solving and theory building. While problem solving has a robust presence in the mathematics curriculum, it is less clear whether theory building does, or should, have such a place. I will report on a curricular design to support a kind of simulation of mathematical theory building. It is based on the notion of a "common structure problem set" (CSPS). This is a small set of mathematical problems with a two-part assignment: I. Solve the problems; and II. Find, articulate, and demonstrate a mathematical structure common to all of them. Some examples will be presented and analyzed. Relations of this construct to earlier ideas in the literature will be presented, in particular to the notion of isomorphic problems, and cognitive transfer. Designing effective instructional enactments of a CSPS is still very much in an experimental stage, and feedback about this would be welcome.

Key words: Theory Building, Mathematical Structure, Mathematical Practices, Transfer

Introduction

Calls for the practices of the discipline to be featured in the teaching and learning of mathematics are a recurring refrain: the Common Core Mathematical Practice Standards (2010); the NCTM Process Standards (2000); the *Adding It Up* strands of mathematical proficiency *(NRC, 2001)*; the EDC Habits of Mind (2011); etc. Mathematical practices refer broadly to the kinds of things one does when doing mathematics, and the above sources do indeed capture important aspects of this. But if one asks a professional mathematician, a disciplinary practitioner, what doing mathematics involves, the first order response often features something absent from all of these sources, illustrated by this quote from Timothy Gowers (2000): "The "two cultures" I wish to discuss will be familiar to all professional mathematicians. . . . I mean the distinction between mathematicians who regard their central aim as being to solve problems, and those who are more concerned with building and understanding theories."

These two modes of mathematical work, *problem solving* and *theory building*, though symbiotic, and not always separated by clear boundaries, are importantly distinct. Are they manifested in the mathematics curriculum? If so, where and how? In fact problem solving has a fairly robust presence, across many grades, and supported by the writings of Polya (1957), Schoenfeld (1994), and others. But theory building? *What activities in the mathematics curriculum could reasonably be construed to be opportunities to learn theory building practices?* That is the central question that animated the work I shall describe here. This question begs for clarification of the underlying concepts, but that clarification itself becomes an important theoretical part of the research.

In the mathematics curriculum theories typically first appear in high school geometry, and in university level courses (calculus, linear algebra, real analysis, abstract algebra, etc.). What such courses generally offer are aspects of a mature, well-articulated theory, and what students are expected to learn is how to manipulate the ideas and techniques of that ready-made theory for problem solving and applications (sometimes routine, sometimes ambitious). What these courses do not (even pretend to) do is have students experience the processes that led to the creation of the fundamental mathematical structures of the theory. In fact it would likely be

impractical to incorporate this goal together with the more applications oriented design of such courses in the same time frame.

Mathematical theories commonly arise when a variety of different problems are solved by methods that seem to have something fundamental in common. By distilling, and articulating that common underlying *mathematical structure*, one can create a conceptual frame that simultaneously resolves, or at least illuminates, all of the problems as different specializations and contextualizations of one, more abstract, problem. This general formulation then has the potential to help solve many more such problems (those that can be modeled by the theory) as well. Thus, while theory building might first seem to be an idea somewhat remote from the school curriculum, we see that it is closely linked to ideas – like mathematical structure, connections, generalization – that are more commonly evoked in the mathematics education literature.

I have thus chosen to formulate and operationalize one notion of theory building activity as *seeking, using, and developing mathematical structure*. This formulation can be viewed as an elaboration of the Common Core mathematical practice – *finding and using mathematical structure*. For example, our system of place value notation for numbers is a (very important) *developed*, or *built* mathematical structure, whereas the Mandelbrodt Set is one *sought* and *found*. One virtue of the above formulation is that it works at many grain sizes, and does not require some grand edifice. For example, when a kindergartner characterizes a repeating pattern as "same, same, different, same, same, different, . . ." she is abstracting a mathematical structure that she could then reconstruct with other materials, sounds, gestures, etc.

Common structure problem sets (CSPS)

I offer here a curricular design that is intended to simulate aspects of the theory building practice of the discipline, as characterized above. The design is conceptually a template that can be applied across different mathematical domains, and I have constructed several instantiations, at different levels, and in different subject areas. While the mathematical construct is well formulated, its instructional enactment, and success in achieving important learning goals, is still very much experimental.

The basic construct is what I call a *common structure problem set* (CSPS). This consists of a collection of (say 4-10) mathematics problems, with a two part assignment: I. Solve the problems; and II. Find, articulate, and demonstrate a mathematical structure that is common to all of these problems. The two parts of the assignment correspond to: (I) problem solving; and (II) theory building. In order for this to authentically approximate the mathematical practice of theory building, it is important that the presence of a structure common to all of the problems not be transparent, or evident on the face of things. Moreover, the problems might be quite varied, for example asking very different kinds of questions about a common structure. Two illustrative examples are given and analyzed below. For each CSPS, I provide an analysis (mathematical, pedagogical, cognitive demand) of each task, and an explication of a (not "the") common mathematical structure.

Links to the literature

The mathematics education literature is replete with references to mathematical structure. For example, the "New Math" reforms featured a Bourbaki style version. Dienes and Jeeves (1965) used the mathematical notion of a group. They designed instruction for 10 year olds to learn the structure of some small groups – the group with two elements, and the Klein four-

group. The group elements were designated by letters, but the children had to discover the law of composition, and group properties. In an interesting and somewhat analogous design, Simpson et al. (2006) describe work with a pre-service secondary teacher who was given an encrypted version of the modular ring $\mathbf{Z}_{99}$, and guided to discover its commutative ring structure.

The concept of a *multiple-solution connecting task* was introduced by Leikin et al. (2007). It refers to a single mathematical task that can be used to exhibit mathematical connections, such as multiple representations of a concept, or links between different concepts in the same domain, or even links between different domains.   A related idea is that of an *interconnecting problem*, developed by Kondratieva (2011). Such a problem (1) allows a simple formulation (without specialized mathematical terms and notions); (2) enables various solutions at both elementary and advance levels; (3) may be solved by various mathematical tools from distinct mathematical branches, and finding multiple solutions; and (4) is used in different grades and courses and can be understood in various contexts. She illustrates these features with the following example: Given a point inside an angle, draw a circle tangent to the sides of the angle and passing through the given point. The notion of "multiple-solution connecting task" is greatly elaborated in the book of Sally and Sally (2007). While these ideas all emphasize mathematical connections, they are distinct from the idea of different problems with common structure, discussed here.

Silver (1979), inspired by Polya's heuristic, "think of a related problem," conducted a study of students' perceptions of relatedness of families of word problems, designed to vary on two dimensions of relatedness: structure; and context. Among his findings, students with high proficiency levels, by a variety of measures, tended to sort problems by structure, while those with lower levels focused more on context. One might then ask whether, reciprocally, nurturing sensitivity to structure could support broader kinds of proficiency.

"Isomorphism" of two problems is a special case of the notion of common structure discussed here. Roughly speaking, two problems A and B are said to be *isomorphic* if there is a correspondence between the elements (objects and operations) of A and B so that a solution process for problem A translates directly into a solution process for problem B (and conversely). Isomorphic problems have the same underlying structure, but common structure (as used here) does not imply isomorphism; it is instead more general. Here is an example of a pair (A,B) of isomorphic problems, taken from Greer and Harel (1998):

A. A straight angle is partitioned into four angles, α_1, α_2, β_1, and β_2, with $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Find the measure of $\alpha_1 + \beta_2$.
B. You and your sister had 180 dollars altogether. Your sister gave me half of what she had, and you gave me half of what you had. How much money do you have left between you?

The isomorphism of these problem pairs is often not seen by problem solvers.

The notion of isomorphic problems seems first to have been taken up mainly in cognitive science, in connection with the question of (cognitive) transfer: To what extent does student understanding of how to solve problem A transfer to the ability to solve an isomorphic problem B, believed to differ from A only in "surface features." For example Simon and Hayes (1976), after initiating subjects to the Towers of Hanoi problem, presented them with a contextualized problem isomorphic to it. In another similar study Siegler (1977) defines two problems to be isomorphic if "they are formally identical but differ in their surface features." He compares children's strategies in a 20-question game, to "guess my number" (among 1 through 24) and, in an isomorphic problem, "guess my letter" (among A through X). In general, transfer effects were found to be weak. Lave (1988) published an influential critique of the theoretical roots and culture of such transfer research. Lobato et al. (2002) further questioned the surface/structure

distinction. They noted that, "a surface feature can present conceptual complexities for students that are more structural in nature than previously understood." They illustrate this with a problem about finding the slope of a line, and another, with common structure, about the steepness of a wheelchair ramp.

Janackova and Janacek (2006) analyzed the different solution strategies used by a high school student on four isomorphic problems in combinatorics. Their problems are in fact equivalent to a subset of the "Pascal" CSPS considered below. But they did not present the student with the task of identifying, articulating, and demonstrating the structure common to all of these problems.

More recently, the CME curriculum on combinatorics (Educational Development Center, 2011) features the notion of isomorphic problems. Moreover, Lin et al. (2011) and others have used isomorphic problems to study analogical reasoning across domains, for example mathematics and physics.

Maher et al. (2010) report on one of the few longitudinal studies of children's development of high level mathematical practices, including development of schemas for connecting structurally related problems. Powell et al. (2009) identified a set of conditions in this project that seemed to shape students' success with connections among the problems. First, students were given *challenging tasks* (in combinatorics) to work on, tasks that were accessible but for which they had no previously developed strategies. Second, students were asked to work on "*strands*" *of problems*, superficially different but which shared the same mathematical structure (in fact were often isomorphic), and researchers encouraged the students to try to relate new problems to previously solved ones. A third feature of the work was that students were given *sufficient time* to explore the problems and to revisit the problems they had explored. Finally, students developed effective *heuristics*, such as first scaling a problem down to an easier version, or generating data and looking for patterns, etc. The authors note also that extended time was important for students to develop skills for detecting connections.

Powell et al. (2009, p. 135) further point out that, ". . . expert problem solvers have and use schemas – or abstract knowledge about the underlying, similar mathematical structure of common classes of problems."   They "can categorize problems into types based on their underlying mathematical structure."   This is consistent with the work of Silver (1979) cited above. Evidently this sensitivity to structure, and the notion of a "strand of problems" (the second condition above) is close to what I am calling a common structure problem set. In fact, the strand illustrated in Powell et al. (2009) is a subset of the "Pascal" CSPS that I present below. What distinguishes this approach from the one that I propose is not in the nature of the mathematical problems, but rather in the charge to the students. With a common structure problem set, the students are explicitly challenged to *identify, articulate, and demonstrate* a mathematical structure common to all of the problems. In other words, what is often left cognitively tacit, and hoped to be assimilated by students, I propose to make an explicit goal of the instruction.

In 1968 Zal Usiskin published, in The Mathematics Teacher, a brief paper, "Six nontrivial equivalent problems."   In fact, his six problems constitute an early example of a common structure problem set. His paper was addressed to teachers, but there was no discussion then of the possible use of such problem sets to challenge students to find and demonstrate common structure.

The Pascal CSPS

1. (Taxi cab geometry). A taxi wants to drive (efficiently) from one corner to another that is 10 blocks north, and 4 blocks east. How many possible routes are there to do this?

2. (Triangular graph) In the (unbounded) triangular array,
 	
                   •
            •   •
         •   •   •
      •   •   •   •
   •   •   •   •   •
•   •   •   •   •   •
 	
.   .   .   continued   .   .   .
 	
connect each dot by an edge to the two nearest dots just below it. At each dot, write the number of "edge-paths" downward to it from the top dot. What is the number in 15th row, the 5th dot from the left?

3. (Walk on the line) On the number line, starting at 0, you are to take 14 steps, each of which is either distance 1 to the right, or distance 1 to the left, and in such a way that you end up at -6. How many ways are there to do this?
4. (Unifix towers) Using 10 white and 4 red unifix cubes, how many different 14-cube towers can you make?
5. (Soccer score progression) The home team won a soccer game 10 to 4. What are all the possible sequences of scoring as the game progressed?
6. (Choosing a team) In a class of 14 students, you need to select a 4-student team. How many different ways are there to do this?
7. (Balls in two bins) What are all ways of putting 14 balls into two bins so that 10 balls are in bin A and 4 balls in bin B.
8. (Cutting a ribbon) You are to cut a 15-inch ribbon into five pieces, each of length a whole number of inches. How many ways are there to do this?
9. (Binomial theorem) In the polynomial $(1 + y)^{14}$, what is the coefficient of y^4?
10. (Higher derivatives of a product) Let $Df = f'$, denote the derivative of a real function f. If f and g are two differentiable functions, then $D^{14}(f \bullet g)$ will be a linear combination of the functions $(D^p f) \bullet (D^{14-p} g)$ $(0 \le p \le 14)$. What is the coefficient of $(D^4 f) \bullet (D^{10} g)$?

Some will quickly recognize that that these problems lead to the binomial coefficient, 14-choose-4 = $14 \bullet 13 \bullet 12 \bullet 11/4! = 7 \bullet 11 \bullet 13 = 1,001$, the number of ways of choosing 4 things from a group of 14 things. Nonetheless, problems 1-8 can be approached without much prior knowledge of such combinatorics. Specifically, the solution space for each of these problems can be modeled by what I shall provisionally call, "binary sequences," i.e. sequences of terms which have only two possible values. More precisely, we define a (14,4)-(X,Y)-sequence to be a sequence of 14 terms, each term being either an X or a Y, and exactly 4 of the terms are Ys. The number of these is 14-choose-4, since it is determined by the 4 (out of the 14) places where the Y

terms appear. I will demonstrate now that the set of such binary sequences is a structure that models the solution space of each of problems 1-8.

1. <u>Taxi cab geometry</u>. The taxi must drive 14 blocks, 4 of them east (E) and 10 of them north (N), so the set of routes is represented by (14,4)-(N,E)-sequences.

2. <u>Triangular graph</u>. An edge path downward from the top dot can be represented by a sequence of right (R) and left (L), since those are the direction choices when descending from any vertex. To arrive at row 15, we need a sequence of length 14 such choices. To arrive at the 5th dot from the left, exactly 4 of the terms must be R. Thus the set of edge paths in question is modeled by (14,4)-(L,R)-sequences. (An alternative perspective is to view the image of problem 2 as like a 135° rotation of the image of problem 1.)

3. <u>Walk on the line</u>. A 14-step walk on the line is modeled by a (-1,+1)-sequence of length 14, with +1 = a step to the right, and -1 = a step to the left. The number at which this walk arrives is simply the sum of the terms of the sequence. In order to arrive at -6 one needs 10 (-1)s and 4 (+1)s. Thus, the solution space here is represented by (14,4)-(-1,+1)-sequences.

4. <u>Unifix towers</u>. Clearly the solution space here is represented, with W = white, and R = red, by (14,4)-(W,R)-sequences.

5. <u>Soccer score progressions</u>. The solution space here is represented by (14,4)-(H,V)-sequences, where H denotes a point scored by the home team, and V a point scored by the visitors.

6. <u>Choosing a team</u>. Labeling the students 1, 2, . . . , 14, the solution space is represented by (14,4)-(0,1)-sequences, where 1 (or 0) in position j signifies that student j is on, (or off) the team.

7. <u>Balls in two bins.</u> If we label the balls 1, 2, . . . , 14, then the solution space is represented by (14,4)-(A,B)-sequences, with (A in position j) signifying (put ball j in bin A), and similarly for B.

8. <u>Cutting a ribbon</u>.   Put inch markers on the 15 inch ribbon. There are 14 of these strictly between the two ends. To cut the ribbon into 5 pieces as in problem 7 is to cut the ribbon at 4 of the above 14 inch-markers. Thus the solution space consists of all ways of choosing 4 cuts out of 14 places, so this is isomorphic to the situation in problem 5, for example.

Note that, even though we see a common structure for (the solution space of) each of these 8 problems, we have not indicated how to count the size (14-choose-4) of these solution spaces. However, the common structure shows that this computation need be done only once, and not separately for each problem.

Problems 9 and 10 require more background knowledge than do problems 1-8. Specifically, problem 9 directly invokes the Binomial Theorem. Here is a treatment that explicitly links the binomial coefficients to binary sequences, as above. First consider a product of n binomials, $P = (a_1+b_1)(a_2+b_2) \cdot \cdot \cdot (a_n+ b_n)$. Multiple use of the distributive law shows that P is the sum of all products $c_1c_2 \cdot \cdot \cdot c_n$ where each c_j is either a_j or b_j. Now suppose that $a_j = a$ and $b_j = b$ for all j, so that $P = (a + b)^n$. Then $c_1c_2 \cdot \cdot \cdot c_n$ above can be viewed as an (n,p)-(a,b)-sequence, where $c_j = b$ for p of the values of j = 1, 2, . . . , n, and then $c_1c_2 \cdot \cdot \cdot c_n = a^{n-p}b^p$. Thus, the coefficient of $a^{n-p}b^p$ in $(a + b)^n$ is the number of (n,p)-(a,b)-sequences. When n = 14 and a = 1, the coefficient of b^4 is the number of (14,4)-(1,b)-sequences.

Problem 10 involves a more sophisticated occurrence of the Binomial Theorem:

(*) $\quad\quad\quad\quad D^n(f \bullet g) \quad\quad = \quad\quad \Sigma_{0 \le p \le n}$ (n-choose-p)$\bullet(D^{n-p}f)\bullet(D^p g)$

This can be proved by induction, using the usual product rule ($n = 1$), plus the Pascal relation for binomial coefficients. However, it naturally tempts one to see if (*) is in fact a special case of the Binomial Theorem. To see that this is indeed the case, see Appendix A. While problem 10 does not directly involve a structure in common with problems 1-8, it is perhaps worth exhibiting as a complement to the other problems to show the widespread and sometimes unexpected manifestations of the Binomial Theorem.

An instructional design for this common structure problem set might reasonably begin with a small subset of problems 1-8, leading to combinatorial formulas for n-choose-p, and then progressively introduce the remaining problems with the suggestion to relate them to prior problems. Problems 9 and 10 of course would first require a treatment of the Binomial Theorem.

The Measure exchange CSPS

1. (Tea & wine) I have a barrel of wine, and you have a cup of green tea. I put a teaspoon of my wine into your cup of tea. Then you take a teaspoon of the mixture in your teacup, and put it back into my wine barrel. Question: Is there now more wine in the teacup than there is tea in the wine barrel, or is it the other way around?
2. (Heads up) I place on the table a collection of pennies. I invite you to randomly select a set of these coins, as many as there were heads showing in the whole group. Next I ask you to turn over each coin in the set that you have chosen. Then I tell you: The number of heads now showing in your group is the same as the number of heads in the complementary group. Question: How do I know this?
3. (Faces up) I blindfold you and then place in front of you a standard deck of 52 playing cards in a single stack. I have placed exactly 13 of the cards face up, wherever I like in the deck. Your challenge, *while still blindfolded*, is to arrange the cards into two stacks so that each stack has the same number of face-up cards.
4. (Triangle medians) In a triangle, the medians from two vertices form two triangles that meet only at the intersection of the medians. How are the areas of these two triangles related? More precisely, let ABC be a triangle. Let A' be the mid-point of AC, B' the mid-point of BC, and D the intersection of AB' and BA'. How are the areas of AA'D and BB'D related?
5. (Trapezoid diagonals) The diagonals a trapezoid divide the trapezoid into four triangles. What is the relation of the areas of the two triangles containing the legs (non parallel sides) of the trapezoid?
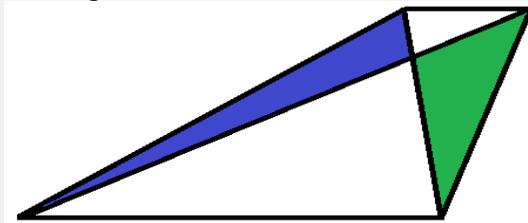
This problem set is unusual in several respects. While problems 4 and 5 fit comfortably in the geometry curriculum, problems 1-3 are essentially puzzles, and it is far from clear that these problems share a common structure. Problem 1 involves comparison of quantities of liquid. Problem 2 (resp. 3) involves comparison of numbers of pennies (resp., cards). And Problems 4 and 5 involve comparisons of areas. Thus each problem involves some species of *measurement*: liquid volume, numbers of cards or pennies, and area. I will argue that the structure these problems have in common is a simple (and self evident) principle of measurement: *If two quantities have equal measure, and you remove from each what they have in common, then what remains of each of them still have equal measure.* Let me formalize this as what I shall call the "Measure Exchange Lemma."

Consider some measurable object M. It might be a volume of liquid, a collection of pennies, a deck of cards, or a plane region. If X is a part of M, let $m(X)$ be its measure (volume, cardinal, area, as the case may be). Let W and W' be subsets of M of the same measure:

(1) $m(W) = m(W')$.   Then: (2) $m(W \backslash W') = m(W' \backslash W)$[1].

Proof that (1) => (2): It follows from (1) that: $m(W \backslash W') = m(W) - m(W \cap W') = m(W') - m(W \cap W') = m(W' \backslash W)$, whence (2).

Now let $T = M \backslash W$ and $T' = M \backslash W'$, the complements in M of W and W', respectively. Then $W \backslash W' = W \cap T'$ and $W' \backslash W = W' \cap T$. So we can restate (2) as:

(3) $m(W \cap T') = m(W \cap T')$.

I will now show that this Measure Exchange Lemma is a mathematical structure common to each of the Measure Exchange Problems.

Application to trapezoid diagonals (#5): Each of the two diagonals of a trapezoid M decomposes M into two triangles, call them T and W for one diagonal and T' and W' for the other, and so that T and T' share a common base, and likewise for W and W'. Since all four triangles have the same height (the distance between the parallel sides of M), it follows that

$$m(T') = m(T) \quad \text{and} \quad m(W') = m(W)$$
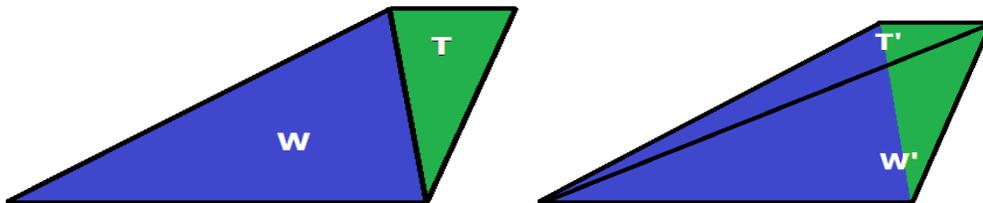
It follows then from (statement (3) of) the Measure Exchange Lemma that $m(W' \cap T) = m(W \cap T')$.

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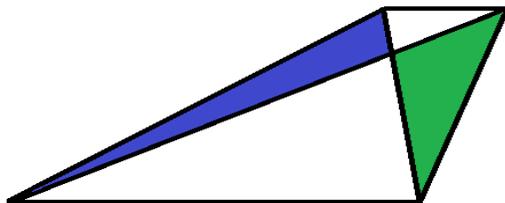
<u>Application to the triangle medians (#4)</u>: This is just an application of #5 to the quadrilateral AA'B'B, that is easily seen to be a trapezoid, with parallel sides AB and A'B'.

<u>Application to the tea and wine problem (#1)</u>: Let M be the combined totality of the tea, T, in the teacup and the wine, W, in the wine barrel. After the two teaspoon exchanges, let T' be the mixture in the teacup, and W' the mixture in the wine barrel. Clearly we have the conditions of the Measure Exchange Lemma, and so we have

m(tea in the wine barrel) = m(W'∩T) = m(W∩T') =   m(wine in the teacup)

<u>Application to the Heads Up problem (#2)</u>: Let M be the collection of pennies on the table, T those with tails up, and W those with heads up. Let W' be the coins that you choose, and T' those you left behind. Again, we have the conditions of the Measure Exchange lemma, and so #(W'∩T) = #(W∩T'). In other words, the number of tails (T) showing in your group (W') = The number of heads (W) showing in what you left behind (T'). After you turn your coins (W') over, the two groups have the same number of heads showing.

<u>Application to the Faces Up in the Deck (#3)</u>: Let M be the deck of cards, W the 13 cards that are face up, and T the complement. While blindfolded, you choose a set W' of any 13 of the cards and put them aside, without any of the cards being turned over; let T' be the complement of W'. The Measure Exchange Lemma tells us that #(W'∩T) = #(W∩T'). In other words, the number of face down cards (T) among your chosen cards (W') =   the number of face up cards (W) among those you did not choose (T').

After turning your chosen cards over there will be the same number of face up cards in each group.

As observed above, problem 4 is easily reduced to problem 5. Also problems 2 and 3 are fairly easily seen to involve common structure (pennies, heads, tails) corresponding to (cards, face up, face down). On the other hand these problems ask very different questions about this structure. In a sense, problem 2 gives you a theorem and asks you to prove it. On the other hand, problem 3 expects you to guess the theorem, and apply it. In this sense, I consider problem 3 to be more challenging than problem 2, especially if it were presented in the absence of problem 2.

I learned Problem 1 from Vladimir Arnold, who said that this kind of problem was presented by Russian parents to young children, prior to their formal mathematical training. He claimed that such children solved the problem more easily, and more simply, than mathematicians. This problem seems at first unlike problems 2 and 3 since it involves continuous rather than discrete measurement. Moreover the analysis above shows that the details of the teaspoon exchanges are mostly irrelevant. In fact one could mix the tea and wine arbitrarily together and then divide the mixture into a teacup and a barrel, and the mathematics of the problem would be unchanged.

This common structure problem set has some attractive features. The problems require very little background, except for some elementary geometry in problems 4 and 5. Moreover, it is far from obvious that they share a common mathematical structure, and it is somewhat subtle to identify and articulate such a structure. In this sense, work on this CSPS provides a rich opportunity to engage in mathematical structure practices. It seems well suited for small group work. Moreover, the puzzle problems (1-3) are interesting to share with friends and family.

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