LESSON 1
Plane Curves

A planar curve is the image of a function from a 1-dimensional space to \( \mathbb{R}^2 \). By a "1-dimensional space" we may either an interval \([a, b]\), the entire real line \( \mathbb{R}^1 \), or a circle \( S^1 \). The function itself is called a parametrization of the curve. In Calculus III you learned that the derivative of such a parameterization (when it exists) is a vector. For the purposes of this class, all parameterizations will be assumed to be differentiable, with non-zero derivative (almost?) everywhere.

Example 1.1. The function \( \psi(t) = (\cos t, \sin t) \), where \( 0 \leq t \leq \pi \), is a parameterization of half of a circle. The derivative of this is \( \frac{d\psi}{dt} = (-\sin t, \cos t) \). Note the change in notation from parentheses to brackets to indicate a shift from points to vectors.

Homework

Problem 1. Occasionally we will describe curves as the set of points for which some equation is true, although in general a parametrization will be much more useful. Find parameterizations of the following:

1. The line \( y = mx + b \)
2. The parabola \( y = x^2 \)
3. The circle \( x^2 + y^2 = 1 \)
4. The ellipse \( \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \)
5. The hyperbola \( \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \)

Here are a few more planar curves that may be new to you.

Problem 2. Consider a curve in \( \mathbb{R}^2 \) whose velocity (tangent vector) never vanishes and intersects all the radial lines from the origin at a constant angle \( \theta_0 \). These are called loxodromes.

1. Determine what the loxodromes with \( \theta_0 = 0 \) and \( \theta_0 = \frac{\pi}{2} \) must be.
2. Show that the logarithmic spirals, \( ae^{bt}(\cos t, \sin t) \), are loxodromes.

Problem 3. A cycloid is a planar curve that follows a point on a circle of radius \( R \) as it rolls along a straight line without slipping.

1. Show that \( R(t - \sin t, 1 - \cos t) \) is a cycloid.
(2) Show that this cycloid hits the $x$-axis at points that are $2\pi R$ apart.

**Problem 4.** The *conchoid of Nicomedes* is given by the quartic (degree 4) equation

$$(x^2 + y^2)(y - b)^2 - R^2 y^2 = 0.$$ 

Descriptively it consists of two curves that are given as points $(x,y)$ whose distance along radial lines of the form $y = tx$ to the line $y = b$ is $R$. (The radial line is simply the line that passes through the origin and $(x,y)$.) Find a parameterization for the conchoid of Nicomedes in terms of $t$.

**Problem 5.** The *cissoid of Diocles* is given by the equation

$$x(x^2 + y^2) = 2Ry^2.$$ 

Let $y = tx$ and find a parameterization in terms of $t$.

**Problem 6.** Let $\alpha_1(t)$ and $\alpha_2(t)$ be parameterization of planar curves so that for each value of $t$, $\alpha_i(t)$ is a point on the line $y = tx$. Then the curve $\alpha(t) = \alpha_1(t) - \alpha_2(t)$ is called a *cissoid*. Show that the conchoid of Nicomedes is a cissoid.

**Problem 7.** Let $q$ be a cissoid where $q_1$ is a circle of radius $R$ centered at $(R,0)$ and $q_2$ the vertical line $x = b$.

1. Show that when $b = 2R$ we obtain the cissoid of Diocles.
2. Show that when $b = \frac{R}{2}$ we obtain the *trisectrix* (*trisector*) of Maclaurin

$$2x(x^2 + y^2) = -R(3x^2 - y^2).$$

3. Show that when $b = R$ we obtain a *strophoid*

$$y^2(R - x) = x^2(x + R).$$
Lesson 2
Speed and Arc Length

Question 2.2. Sketch the following curves. What is the difference between them?

1. \((\cos t, \sin t)\)
2. \((\cos 2t, \sin 2t)\)
3. \((\cos t^2, \sin t^2)\)

The speed \(s_\alpha(t)\) of a parameterization \(\alpha(t)\) is the rate of change of distance travelled, per unit of time. That is,

\[
s_\alpha(t) = \lim_{\Delta t \to 0} \frac{|\alpha(t + \Delta t) - \alpha(t)|}{\Delta t}
\]

Note that this is a vector calculation: The expression \(\alpha(t + \Delta t) - \alpha(t)\) is a vector from \(\alpha(t)\) to \(\alpha(t + \Delta t)\), and the bars indicate that you are to take the magnitude of this vector.

There is another way to think about speed: it is the magnitude of the derivative vector. That is,

\[
s_\alpha(t) = |\alpha'(t)|
\]

This equivalence is not completely trivial. It relies on the fact that both limits and division by scalars commute with taking the magnitude of a vector.

Keep in mind that vector operations are often useful to compute magnitude:

\[
s_\alpha(t) = |\alpha'(t)| = \sqrt{\alpha'(t) \cdot \alpha'(t)}
\]

In particular, if we know \(\alpha\) has constant unit speed, then

\[
\alpha'(t) \cdot \alpha'(t) = 1
\]

We will make frequent use of this later.

Question 2.3. Calculate the speed for each curve in the question above.

Definition 2.4. A parameterized curve is regular if its derivative is non-zero everywhere.

Question 2.5. Which curves in the first question are regular?

Closely related to speed is arc length. To calculate the length of the curve \(\alpha(t)\) from \(t = a\) to \(t = b\) we first choose points \(\{t_i\}\) in the interval \([a, b]\) that are \(\Delta t\) apart. For each \(i\), we estimate the length between \(\alpha(t_i)\) and \(\alpha(t_{i+1}) = \alpha(t_i + \Delta t)\) by using the linear distance between
these points. Then add, and take the limit as $\Delta t$ goes to zero. Thus, the total length is a Riemann sum:

$$\lim_{\Delta t \to 0} \sum_{i=0}^{n} |\alpha(t_{i+1}) - \alpha(t_i)| = \lim_{\Delta t \to 0} \sum_{i=0}^{n} |\alpha(t_i + \Delta t) - \alpha(t_i)|$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n} \frac{|\alpha(t_i + \Delta t) - \alpha(t_i)|}{\Delta t} \Delta t$$

$$= \lim_{\Delta t \to 0} \sum_{i=0}^{n} \frac{|\alpha(t_i + \Delta t) - \alpha(t_i)|}{\Delta t} \Delta t$$

$$\approx \lim_{\Delta t \to 0} \sum_{i=0}^{n} s_{\alpha}(t_i) \Delta t$$

$$= \int_{a}^{b} s_{\alpha}(t) \, dt$$

In other words, we calculate arclength by integrating speed.

**Question 2.6.** Calculate the arc length of each curve in the first question, from $t = 0$ to $t = \pi$.

We will see later that parameterizations with unit speed are particularly useful (although in practice very difficult to find!). Note that if $\alpha$ is such a parameterization, then the length of $\alpha$, between $t = a$ and $t = b$, is

$$\int_{a}^{b} s_{\alpha}(t) \, dt = \int_{a}^{b} dt = b - a$$

Hence, $\alpha$ takes an interval of length $b - a$ on the real line to a curve of length $b - a$. For this reason, such a function is sometimes called a parameterization with respect to arclength.

Now let’s examine two of the examples from the first question. Both $\alpha(t) = (\cos t, \sin t)$ and $\beta(t) = (\cos 2t, \sin 2t)$ are parameterizations of the circle. The only difference between them is that the second one traverses the circle twice as fast.

If we let $f(t) = 2t$, then observe that $\beta(t) = \alpha(f(t))$. In general, the substitution of $f(t)$ for $t$ doesn’t change the curve, but only the speed along which you traverse it. The new speed can be expressed in terms of the old by the chain rule:

$$s_{\beta}(t) = |\beta'(t)| = |\alpha'(f(t))f'(t)| = f'(t)s_{\alpha}(f(t))$$
In other words, the speed of $\beta$ at some point of the curve is the speed of $\alpha$ at the same point, times $f'(t)$. It follows immediately that if $\alpha$ is regular, and $f'(t) > 0$ for all $t$, then $\beta$ is regular.

**Definition 2.7.** Suppose $\alpha(t)$ is a regular parameterization of some curve $C$, and $f(t)$ is a real valued function with $f'(t) > 0$. The we say $\beta(t) = \alpha(f(t))$ is a reparameterization of $C$.

As was stated earlier, parameterizations with unit speed (i.e. parameterizations with respect to arclength) are particularly useful. As we will see in the next theorem, such parameterizations can, at least theoretically, always be found.

**Theorem 2.8.** Suppose $\alpha(t)$ is a regular curve, defined on $[a,b]$. Then there exists a function $f(s)$ such that the reparameterization $\alpha(f(s))$ has unit speed.

**Proof.** First, let the function $g(t)$ measure the arclength along $\alpha$, from $a$ to $t$. That is,

$$g(t) = \int_{a}^{t} |\alpha'(\tau)| \, d\tau$$

Said another way, $g(t)$ is defined to be the anti-derivative of the speed of $\alpha(t)$. It follows that the derivative of $g(t)$ is just the speed of $\alpha(t)$, i.e. $g'(t) = |\alpha'(t)|$. Since $\alpha(t)$ is assumed to be regular, it follows that

$$g'(t) = |\alpha'(t)| > 0$$

It now follows from the *Inverse Function Theorem* that the function $g$ has an inverse, which we will call $f$. In other words, $f$ is the function such that

$$f(g(t)) = t.$$ 

Differentiating both sides yields:

$$f'(g(t))g'(t) = 1,$$

or

$$f'(g(t)) = \frac{1}{g'(t)} = \frac{1}{|\alpha'(t)|}.$$
Let \( s = g(t) \) (so that \( f(s) = t \)). We are now prepared to compute the speed of the reparameterization \( \beta(s) = \alpha(f(s)) \):

\[
\begin{align*}
\beta'(s) &= \frac{d}{ds} \beta(s) \\
&= \frac{d}{dt} \beta(t) \frac{dt}{ds} \\
&= \frac{d}{dt} \alpha(f(g(t))) \frac{dt}{ds} \\
&= \frac{d}{dt} \alpha(t) \frac{dt}{ds} \\
&= \frac{1}{|\alpha'(t)|} \frac{dt}{ds} \\
&= \frac{1}{|\alpha'(t)|} \frac{1}{|\alpha'(t)|} \\
&= 1
\end{align*}
\]

It’s a little difficult to unpack the reparameterization found in the previous theorem. The function \( g(t) \) is the arclength between \( t = a \) and \( t = t \). The inverse function, \( f(t) \), thus takes a particular length \( s \) and tells you the parameter \( t \) you need to plug in to \( \alpha(t) \) to be \( s \) units of arclength along the curve. Thus, the composition \( \alpha(f(s)) \) takes a length \( s \) and returns coordinates of the point of the curve that is \( s \) units of arclength along the curve from the initial point. This is precisely what a parameterization with respect to arclength should do.

**Question 2.9.** What is the domain of the reparameterization of \( \alpha \) found in the previous theorem?

**Homework**

**Problem 8.** Find a parameterization with respect to arclength for the line segment between \((1, 1)\) and \((3, 4)\).

**Problem 9.** Let \( f(x) = x^{\frac{3}{2}} \).

1. Find a parameterization for the graph of \( y = f(x) \).
2. Find the speed function for the parameterization you found.
3. Compute the arclength between \( x = 0 \) and \( x = 2 \).
4. Find a reparameterization with respect to arclength between \( x = 0 \) and \( x = 2 \).

**Problem 10.** Compute the arclength of

1. The graph of \( y = x^2 \), between \( x = 0 \) and \( x = 2 \).
(2) the logarithmic spiral
\[ ae^{bt}(\cos t, \sin t). \]

(3) the spiral of Archimedes
\[ (a + bt)(\cos t, \sin t). \]
Question 3.10. Sketch the parameterized curve \((R \cos t, R \sin t)\). Then find a parameterization of this curve with respect to arclength.

Suppose \(\alpha(s)\) is a regular parameterized curve with unit speed. Then \(\alpha'(s)\) is a unit tangent vector, which we will denote \(T_\alpha(s)\).

Question 3.11. Find \(T_\alpha(s)\), where \(\alpha(s)\) is the unit speed parameterization you found above for the circle of radius \(R\).

As \(T_\alpha(s)\) is a unit vector, we have
\[
T_\alpha(s) \cdot T_\alpha(s) = 1
\]
Differentiating both sides, and using the product rule (which works for dot products!), we get
\[
T'_\alpha(s) \cdot T_\alpha(s) + T_\alpha(s) \cdot T'_\alpha(s) = 0.
\]
Since the dot product is commutative, this simplifies to
\[
2T'_\alpha(s) \cdot T_\alpha(s) = 0
\]
and thus
\[
T'_\alpha(s) \cdot T_\alpha(s) = 0.
\]
We conclude that the derivative of \(T_\alpha(s)\) is a vector that is always perpendicular to \(T_\alpha(s)\). In general, however, it is not a unit vector. We will see that its magnitude says much about the curve \(\alpha\).

Suppose \(T_\alpha(s) = (a(s), b(s))\). Let \(N_\alpha(s)\) be the vector \((-b(s), a(s))\). Thus, \(N_\alpha(s)\) is the unit vector obtained from \(T_\alpha(s)\) by rotating 90 degrees counter-clockwise. Since \(T_\alpha(s)\) is tangent to the curve, it follows that \(N_\alpha(s)\) is a unit normal vector.

Note that since \(T'_\alpha(s)\) and \(N_\alpha(s)\) are both perpendicular to \(T_\alpha(s)\), they either point in the same direction or in opposite directions. Thus, it makes sense to define:

Definition 3.12. The curvature of the unit speed parameterized curve \(\alpha(s)\) is the function \(\kappa(s)\) such that
\[
T'_\alpha(s) = \kappa(s)N_\alpha(s)
\]
When \(T'_\alpha(s)\) and \(N_\alpha(s)\) point in the same direction, \(\kappa_\alpha(s) = |T'_\alpha(s)|\). When they point in opposite directions, \(\kappa_\alpha(s) = -|T'_\alpha(s)|\). Thus, the curvature of \(\alpha\) measures the rate at which \(T_\alpha(s)\) turns towards \(N_\alpha(s)\). When the curvature is positive the curve is turning counter-clockwise, and when it is negative it is turning clockwise.
**Question 3.13.** Suppose $V$ and $W$ are fixed vectors, where $|W| = 1$. Show that the curvature of the line $V + Ws$ is zero.

We now prove the converse:

**Theorem 3.14.** If $\alpha(t)$ has zero curvature then it is a line.

*Proof.* If the curvature is zero, then $T_\alpha'(s) = 0$. Thus, $T_\alpha(s) = \alpha'(s)$ is a constant unit vector, $W$. Thus $\alpha(s) = V + Ws$, for some constant vector $V$. \hfill $\Box$

**Question 3.15.** Find the curvature of the curves $(R \cos t, R \sin t)$ and $(R \cos t, -R \sin t)$.

**Lemma 3.16.** Suppose $\alpha(s)$ is a unit speed parameterized curve. Then

$$N_\alpha''(s) = -\kappa_\alpha(s)T_\alpha(s).$$

*Proof.* As $N_\alpha(s)$ is a unit vector,

$$N_\alpha(s) \cdot N_\alpha(s) = 1.$$ Differentiating both sides leads to the equation

$$N_\alpha'(s) \cdot N_\alpha(s) = 0$$

and thus $N_\alpha'(s)$ is perpendicular to $N_\alpha(s)$. It must therefore be parallel to $T_\alpha(s)$, and hence $N_\alpha'(s) = f(s)T_\alpha(s)$ for some function $f(s)$. We now determine what this mystery function $f(s)$ is.

The fact that $T_\alpha(s)$ and $N_\alpha(s)$ are perpendicular implies $T_\alpha(s) \cdot N_\alpha(s) = 0$.

Differentiating both sides of this equation gives us

$$T_\alpha'(s) \cdot N_\alpha(s) + T_\alpha(s) \cdot N_\alpha'(s) = 0,$$

and thus

$$T_\alpha(s) \cdot N_\alpha'(s) = -T_\alpha'(s) \cdot N_\alpha(s).$$

We now have all of the ingredients to complete the proof:

$$f(s) = T_\alpha(s) \cdot f(s)T_\alpha(s) = T_\alpha(s) \cdot N_\alpha'(s) = -T_\alpha'(s) \cdot N_\alpha(s) = -\kappa_\alpha(s).$$ \hfill $\Box$

We are now ready to establish the converse of the previous question:

**Theorem 3.17.** If $\alpha(s)$ has constant curvature $c$, then it is a circle of radius $\frac{1}{c}$. 

Proof. Suppose $\kappa_\alpha(s) = c \neq 0$. Let $\beta(s) = \alpha(s) + \frac{1}{c}N_\alpha(s)$. Note that if $\alpha(s)$ were in fact a circle, then $\beta(s)$ would always point to its center. Since $\beta(s)$ in this case is constant, it would follow that $\beta'(s) = 0$. We wish to show that this is always the case.

We compute, and use the previous lemma:

$$\beta'(s) = \alpha'(s) + \frac{1}{c}N'_\alpha(s)$$

$$= T_\alpha(s) - \frac{1}{c}cT_\alpha(s)$$

$$= 0.$$

Now that we know $\beta(s)$ is a point, we will complete the proof by showing that $\alpha(s)$ maintains a constant distance of $\frac{1}{c}$ from it:

$$|\alpha(s) - \beta(s)| = \left|\alpha(s) - \left(\alpha(s) + \frac{1}{c}N_\alpha(s)\right)\right|$$

$$= \left|\frac{1}{c}N_\alpha(s)\right|$$

$$= \frac{1}{c}.$$

Note that knowing the curvature of $\alpha$ is a constant $c$ does not tell us which circle of radius $\frac{1}{c}$ it is. However, any two such circles are related by a rigid motion of the plane. This hints at our ultimate goal: showing that the curvature function $\kappa(s)$ completely determines the curve $\alpha(s)$, up to rigid motions of the plane. We will show this later.

\[\Box\]

Homework

Problem 11. Let $M$ be a rigid motion of the plane (i.e. some combination of translation and rotation), and let $\alpha$ be a unit speed plane curve. Show that

$$\kappa_{M(\alpha)}(s) = \kappa_\alpha(s).$$

(Hint: You may assume that for any vector valued function $V(s)$, $(M(V(s))' = M(V'(s)).)$

Problem 12. Show that applying a reflection in a straight line to a plane curve changes the sign of its curvature.
LESSON 4
Computing curvature with non-unit speed curves

In the previous lesson we saw how to compute the curvature of unit-speed curves. Unfortunately, most curves that we encounter do not have convenient unit-speed parameterizations. In this lesson we will see how to deal with those.

Suppose \( \alpha(t) \) is a regular parameterization. As we have seen in previous results, we can replace \( t \) with a function of the arclength \( s \) to get a unit-speed reparameterization \( \alpha(s) \). Henceforth, we will think of \( t \) as a function of \( s \). This is just the inverse function that you get when you think of \( s \) as a function of \( t \) via:

\[
\frac{ds}{dt} = \frac{1}{|\alpha'(t)|},
\]

i.e. the parameter \( s \) is the anti-derivative of the speed \( |\alpha'(t)| \). It follows that \( \frac{dt}{ds} = \frac{1}{|\alpha'(t)|} \), and thus

\[
\frac{dt}{ds} \frac{ds}{dt} = 1.
\]

We now compute. The unit tangent vector to \( \alpha \) is given by the chain rule:

\[
T_\alpha(s) = \frac{d\alpha'(s)}{ds} = \frac{d\alpha(t)}{dt} \frac{dt}{ds} = \alpha'(t) \frac{dt}{ds},
\]

The curvature \( \kappa_\alpha \) is then given by

\[
\kappa_\alpha(s) = \frac{dT_\alpha(s)}{ds} \cdot N_\alpha(s)
= \frac{dT_\alpha(s)}{ds} \cdot R_{90}(T_\alpha(s))
= \frac{d}{ds} \left( \alpha'(t) \frac{dt}{ds} \right) \cdot R_{90} \left( \alpha'(t) \frac{dt}{ds} \right)
= \left( \alpha''(t) \left( \frac{dt}{ds} \right)^2 + \alpha'(t) \frac{d^2t}{ds^2} \right) \cdot R_{90} \left( \alpha'(t) \frac{dt}{ds} \right)
= \left( \frac{dt}{ds} \right)^3 \alpha''(t) \cdot R_{90}(\alpha'(t)) + \frac{d^2t}{ds^2} \frac{dt}{ds} \alpha'(t) \cdot R_{90}(\alpha'(t))
= \left( \frac{dt}{ds} \right)^3 \alpha''(t) \cdot R_{90}(\alpha'(t)) + 0
= \frac{\alpha''(t) \cdot R_{90}(\alpha'(t))}{|\alpha'(t)|^3}
\]
Question 4.18. Compute the curvature of the graph of $y = x^2$.

Definition 4.19. Let $\alpha(s)$ be a unit speed plane curve with nowhere zero curvature. The center of curvature $\epsilon_\alpha(s)$ of $\alpha$ at the point $\alpha(s)$ is the curve

$$\epsilon_\alpha(s) = \alpha(s) + \frac{1}{\kappa_\alpha(s)} N_\alpha(s).$$

For each fixed $s$, the circle with center at $\epsilon_\alpha(s)$ and radius $\frac{1}{\kappa_\alpha(s)}$ is called the osculating circle to $\alpha(s)$.

Note that when $\alpha$ is a circle, $\epsilon_\alpha(s)$ always points to its center, and thus it’s osculating circle is always just $\alpha$.

Since the curvature of a circle is the reciprocal of its radius, it follows that the osculating circle to $\alpha(s)$ has the same curvature as $\alpha(s)$ does at that point. As the tangent to a circle is perpendicular to its radius, and the radius of the osculating circle through $\alpha(s)$ is in the same direction as $N_\alpha(s)$, it follows that the tangent to the osculating circle at that point is in the same direction as $T_\alpha(s)$. These two facts tell us that the osculating circle is the best-fit tangent circle to the curve $\alpha$. This gives us a new interpretation of curvature: it is the reciprocal of the radius of the best-fit tangent circle.

Homework

Problem 13. Compute the curvature of $(\cos^3 t, \sin^3 t)$.

Problem 14. Let $f(x)$ be a real-valued function with a critical point at $x_0$, i.e. $f''(x_0) = 0$. Show that the curvature of its graph at $(x_0, f(x_0))$ is $f'''(x_0)$.

Problem 15. Let $\beta(s)$ be a unit speed reparameterization of the logarithmic spiral $ae^{bt}(\cos t, \sin t)$. Show that the curvature of $\beta(s)$ is $\frac{b}{s}$.

Problem 16. Let $\alpha(s)$ be a unit speed, regular plane curve. Let $\lambda$ be some constant such that $|\lambda \kappa_\alpha(s)| < 1$ for all values of $s$. Then the $\lambda$-parallel curve of $\alpha$ is defined by

$$\tau(s) = \alpha(s) + \lambda N_\alpha(s).$$

(1) Show that every parallel curve is regular.

(2) Show that $\kappa_\tau(s) = \frac{\kappa_\alpha(s)}{1 - \lambda \kappa_\alpha(s)}$. 
Problem 17. As we vary $s$, the center of curvature $\epsilon_\alpha(s)$ of a curve $\alpha(s)$ moves. In this way we may regard $\epsilon_\alpha(s)$ as a parameterized curve, called the evolute of $\alpha$. (Technically, the evolute is the unit-speed reparameterization of this curve.) Assume that $\kappa'_\alpha(s) > 0$ for all $t$.

(1) Show that the arc-length of $\epsilon_\alpha(s)$ is $u_0 - \frac{1}{\kappa_\alpha(s)}$, where $u_0$ is a constant.

(2) Calculate the curvature of $\epsilon_\alpha(s)$.

(3) Show that the evolute of a cycloid is a cycloid.

Problem 18. A string of length $l$ is attached to the point $\alpha(0)$ of a unit speed curve $\alpha$.

(1) Show that when the string is wound onto the curve while being kept taut, its endpoint traces out the curve

$$\iota(s) = \alpha(s) + (l - s)T_\alpha(s),$$

where $0 < s < l$. The curve $\iota(s)$ is called the involute of $\alpha(s)$. (Technically, the involute is the unit speed reparameterization of $\iota(s)$.)

(2) Assume $\kappa_\alpha(s) > 0$ for all $s$ and show that $\kappa_\iota(s) = \frac{1}{l-s}$

Problem 19. Let $\alpha(s)$ be a regular plane curve. Show that

(1) The involute of the evolute is a parallel curve of $\alpha$.

(2) The evolute of the involute of $\alpha$ is $\alpha$. 
LESSON 5
Rotation Index

In this section, we see yet another way to think about the curvature of plane curves. This formulation will be useful when we get to the proof that all curves are determined (up to rigid motion) by their curvature.

**Theorem 5.20.** Let \( v \) be a fixed unit vector. Let \( \alpha(s) \) be a unit speed plane curve. Let \( \phi(s) \) denote the angle that one must rotate \( v \) counterclockwise to get \( T_\alpha(s) \). Then

\[
\kappa_\alpha(s) = \phi'(s).
\]

**Proof.** Let \( u = R_{90}(v) \), i.e. the vector we get when we rotate \( v \) 90 degrees counter-clockwise. Then

\[
T_\alpha(s) = \cos \phi(s)v + \sin \phi(s)u.
\]

Thus, \( N_\alpha(s) = R_{90}(T_\alpha(s)) = -\sin \phi(s)v + \cos \phi(s)u \).

Differentiating \( T_\alpha(s) \), we get:

\[
T'_\alpha(s) = -\sin \phi(s)\phi'(s)v + \cos \phi(s)\phi'(s)u = \phi'(s)N_\alpha(s),
\]

and thus \( \kappa_\alpha(s) = \phi'(s) \). \( \square \)

We are now prepared to prove our main result for this section:

**Theorem 5.21.** Let \( v \) be a fixed unit vector, and \( p \) a fixed point. For any smooth function \( \kappa : [a, b] \to \mathbb{R} \) there is a unique curve \( \alpha(s) \) such that \( \alpha(0) = p \), \( T_\alpha(0) = v \), and \( \kappa_\alpha(s) = \kappa(s) \).

**Proof.** Since the derivative of the angle function gives curvature, we can recover the angle that the tangents to the desired curve must make with \( v \) by taking an anti-derivative:

\[
\phi(s) = \int_a^b \kappa(s) \, ds
\]

The desired tangent vector \( T_\alpha(s) \) must then be

\[
T_\alpha(s) = \cos \phi(s)v + \sin \phi(s)u,
\]

where \( u = R_{90}(v) \).

Since we want \( T_\alpha(s) \) to be the derivative of \( \alpha(s) \), we will want to define \( \alpha(s) \) to be some anti-derivative of \( T_\alpha(s) \). Any two such anti-derivatives will differ by a constant. To determine which constant to pick, we write the point \( p \) in the \((v, u)\) basis: Let \( x_0 \) and \( y_0 \) be the
numbers such that \( p = x_0v + y_0u \). Now, let \( f(s) \) be the antiderivative of \( \cos \phi(s) \) such that \( f(0) = x_0 \), and let \( g(s) \) be the antiderivative of \( \sin \phi(s) \) such that \( g(0) = y_0 \). Finally, we define

\[
\alpha(s) = f(s)v + g(s)u.
\]

We now check that the curve \( \alpha(s) \) has the desired properties:

\[
\alpha(0) = f(0)v + g(0)u = x_0v + y_0u = p.
\]

Furthermore,

\[
T_\alpha(s) = \alpha'(s) = (f(s)v + g(s)u)' = f'(s)v + g'(s)u = \cos \phi(s)v + \sin \phi(s)u.
\]

Thus, \( T_\alpha(0) = \cos \phi(0)v + \sin \phi(0)u = v \).

Finally, \( \kappa_\alpha(s) = \phi'(s) = \kappa(s) \), as desired. \( \square \)

**Question 5.22.** Mimic the steps in the above proof to get a different proof that a curve with constant curvature \( c \) must be a circle of radius \( \frac{1}{c} \).

**Question 5.23.** Find a curve whose curvature is \( \frac{1}{1 + s^2} \).

In the previous theorem, the integral of the curvature appears naturally. We now make some observations about the value of this integral for closed curves.

**Definition 5.24.** A parameterized curve \( \alpha : [a, b] \to \mathbb{R}^2 \) is closed if \( \alpha(a) = \alpha(b) \) and \( T_\alpha(a) = T_\alpha(b) \).

We can now state an interest little result:

**Theorem 5.25.** Suppose \( \alpha : [a, b] \to \mathbb{R}^2 \) is a closed curve, parameterized with respect to arclength. Then the total curvature of \( \alpha \) is an integer multiple of \( 2\pi \). That is, for some \( n \in \mathbb{Z} \),

\[
\int_a^b \kappa_\alpha(s) \, ds = 2\pi n.
\]
Proof. We have previously established the following:

\[ \int_a^b \kappa_\alpha(s) \, ds = \int_a^b \phi'(s) \, ds = \phi(b) - \phi(a) \]

Now recall that \( \phi(a) \) and \( \phi(b) \) are the angles \( T_\alpha(a) \) and \( T_\alpha(b) \) make with a fixed unit vector \( \mathbf{v} \). However, by assumption \( \alpha \) is closed, so \( T_\alpha(a) = T_\alpha(b) \), and hence \( \phi(b) - \phi(a) \) is an integer multiple of \( 2\pi \). \( \square \)

Note that for polygons, this result is a generalization of the fact that the sum of the exterior angles is always equal to \( 2\pi \).

Definition 5.26. The number \( n \) that appears in the previous theorem is called the rotation index of \( \alpha \).

Question 5.27. Draw pictures to guess the rotation index of

1. a circle
2. A figure-8
3. The graph of \( r = 2 + \cos \frac{\theta}{2} \), where \( 0 \leq \theta \leq 4\pi \)

Definition 5.28. A closed curve \( \alpha : [a, b] \to \mathbb{R}^2 \) is simple if \( \alpha(s) = \alpha(t) \) implies \( s = t \) or \( \{s, t\} = \{a, b\} \).

The following is apparently due to Ostrowski, although the proof we give here is due to Hopf, and paraphrased from the lectures notes of Peter Petersen.

Theorem 5.29. A simple closed curve has rotation index \( \pm 1 \).

Proof. (Hopf, 1935) We assume that \( \alpha : [0, l] \to \mathbb{R}^2 \) is a simple closed curve parameterized by arclength. Moreover, after possibly rotating and translating the curve we’ll assume that \( \alpha(0) = (0, 0) \), \( T_\alpha(0) = (\pm 1, 0) \), and the \( y \)-coordinate of every point of \( \alpha \) is non-negative. We define the following family of unit vectors on the triangle in the \((s, t)\)-plane where \( 0 \leq s \leq t \leq l \).

\[
T(s, t) = \begin{cases} 
T_\alpha(s) & s = t, \\
-T_\alpha(0) & s = 0, t = l, \\
\frac{a(t) - a(s)}{|a(t) - a(s)|} & \text{for all other } s < t.
\end{cases}
\]

Since the curve is simple, closed, and smooth this will yield a well-defined function whose values are always unit vectors. If we select any simple path in this triangle from \((0, 0)\) to \((l, l)\) then \( T(s, t) \) will wind around the unit circle and end up where it began, as \( T(0, 0) = T(l, l) \). Moreover, if we make a slight change in this path it will wind around the same number of times. Along the diagonal the number of windings is the rotation index of the curve \( \alpha \).
Assume first that $T(0) = \langle 1, 0 \rangle$. Starting at $(0, 0)$ in the $(s, t)$-plane, and moving to $(0, l)$ along the $t$-axis, $T(0, t)$ rotates $\pi$. Then moving along the upper edge of the triangle from $(0, l)$ to $(l, l)$, $T(s, l)$ rotates another $\pi$. Hence, the total rotation of $T(s, t)$ along the left and upper edges of the triangle is $2\pi$. Continuously deforming this path to the diagonal doesn’t change the total rotation of $T(s, t)$, and yet along the diagonal the total rotation is the same as that of $\alpha$.

When instead $T(0) = \langle -1, 0 \rangle$ a similar argument shows the rotation index is $-1$.

**Homework**

**Problem 20.** By directly calculating its total curvature, show that the rotation index of every circle of radius $R$ is $\pm 1$.

**Problem 21.** Show that the rotation index of $\alpha(t) = (\cos t, \sin 2t), 0 \leq t \leq 2\pi$, is zero. (Hint: show that for all $t \in [0, \pi]$, $\kappa_\alpha(t) = -\kappa_\alpha(t + \pi)$.)
LESSON 6
The Four Vertex Theorem

We conclude our study of planar curves with a famous theorem.

Definition 6.30. A simple closed curve \( \alpha(s) \) is convex if for each \( p, q \), the line segment connecting \( \alpha(p) \) to \( \alpha(q) \) only meets \( \alpha \) in its endpoints.

Definition 6.31. Suppose \( \alpha(s) \) is a simple closed curve. A vertex of \( \alpha \) is a local extremum for the function \( \kappa_\alpha(s) \).

It should be obvious that every simple closed curve has at least two vertices, since the function \( \kappa(s) \) will always have a global maximum and a global minimum. The surprise is that if the curve is convex, then there must be at least two more vertices.

Theorem 6.32. Every simple closed convex curve has at least four vertices.

Proof. Let \( \alpha : [a, b] \rightarrow \mathbb{R}^2 \) be a simple closed convex curve parameterized by arclength. Let \( p \) and \( q \) be points of \( \alpha \) where \( \kappa_\alpha(s) \) has a global maximum and minimum, respectively. By perhaps translating and rotating \( \alpha \), we may assume that both \( p \) and \( q \) lie on the \( x \)-axis. (You showed that rigid motions don’t change the curvature function, and hence the number of vertices have not changed).

We now do some integration by parts:

\[
\int_a^b \kappa_\alpha'(s)\alpha(s) \, ds = \left. \kappa_\alpha(s)\alpha(s) \right|_a^b - \int_a^b \kappa_\alpha(s)\alpha'(s) \, ds
\]

\[
= 0 - \int_a^b \kappa_\alpha(s)T_\alpha(s) \, ds
\]

\[
= \int_a^b N_\alpha'(s) \, ds
\]

\[
= N_\alpha(b) - N_\alpha(a)
\]

\[
= \langle 0, 0 \rangle
\]

If \( \alpha(s) = \langle x(s), y(s) \rangle \), then we can write this equality in components as

\[
\int_a^b \kappa_\alpha'(s)\alpha(s) \, ds = \left\langle \int_a^b \kappa_\alpha'(s)x(s) \, ds, \int_a^b \kappa_\alpha'(s)y(s) \, ds \right\rangle = \langle 0, 0 \rangle
\]
This is really two assertions in one. We will only need the second:

\[
\int_{a}^{b} \kappa'_{\alpha}(s)y(s) \, ds = 0
\]

We will assume \( \alpha \) is oriented counter-clockwise, and \( p \) is to the left of \( q \) on the \( x \)-axis. By convexity, the \( x \)-axis divides \( \alpha \) into two curves; one above the \( x \)-axis, and one below. Let \( \alpha^+ \) denote the subset of \( \alpha \) above the \( x \)-axis, and \( \alpha^- \) the subset below. Then on \( \alpha^+ \) the function \( y(s) > 0 \), and on \( \alpha^- \) the function \( y(s) < 0 \).

Since \( q \) is a minimum for \( \kappa_{\alpha}(s) \) and \( p \) is a maximum, as we traverse from \( q \) to \( p \) along \( \alpha^+ \) the function \( \kappa_{\alpha}(s) \) will be strictly increasing, for otherwise \( \alpha \) would have another vertex there. Hence, on \( \alpha^+ \) the function \( \kappa'(s) \) is strictly positive. We conclude the function \( \kappa'_{\alpha}(s)y(s) \), and hence its integral, must be strictly positive on \( \alpha^+ \). However, by similar logic \( \kappa'_{\alpha}(s) \) is strictly negative on \( \alpha^- \), and hence again the integral of \( \kappa'_{\alpha}(s)y(s) \) must be strictly positive on \( \alpha^- \). We have thus contradicted the fact that the integral of \( \kappa'_{\alpha}(s)y(s) \) is zero on all of \( \alpha \).

We conclude that somewhere along \( \alpha^+ \) or \( \alpha^- \) the curve \( \alpha \) has a vertex, where the sign of \( \kappa'_{\alpha}(s) \) changes. However, there must be an even number of such sign changes total, so somewhere else there must be an additional vertex. \( \square \)

Homework

**Problem 22.** Consider the ellipse parameterized by

\[(a \cos \theta, b \sin \theta), \quad 0 \leq \theta \leq 2\pi\]

(1) Show that its vertices are the points \((\pm a, 0)\) and \((0, \pm b)\).

(2) Calculate its curvature at each of those points.
LESSON 7
Space curves

We now make the transition from two dimensional parameterized curves to three dimensions. We begin with the following observation. Suppose \( \alpha : [a, b] \to \mathbb{R}^3 \) is a unit speed parameterized curve. As before, being unit speed means \( |\alpha'(s)| = 1 \), and in this case the unit tangent vector is \( T_{\alpha}(s) = \alpha'(s) \). Hence,

\[
1 = T_{\alpha}(s) \cdot T_{\alpha}(s).
\]

Differentiating both sides gives

\[
0 = T'_{\alpha}(s) \cdot T_{\alpha}(s) + T_{\alpha}(s) \cdot T'_{\alpha}(s) = 2T'_{\alpha}(s) \cdot T_{\alpha}(s)
\]

and thus, as in the planar case, \( T_{\alpha}(s) \) is perpendicular to \( T'_{\alpha}(s) \). In the planar case we found a canonical choice of normal vector by defining \( N_{\alpha}(s) = R_{90}(T_{\alpha}(s)) \), and then defined \( \kappa_{\alpha}(s) \) by comparing the length of \( T'_{\alpha}(s) \) to that of \( N_{\alpha}(s) \). In three dimensions, there is no such obvious choice for \( N_{\alpha}(s) \). Hence, we use the 2D equation \( T'_{\alpha}(s) = \kappa(s)N_{\alpha}(s) \) as a motivation for defining \( N_{\alpha}(s) \) to be the unit vector that points in the same direction as \( T'_{\alpha}(s) \). With this choice, it again makes sense to define \( \kappa(s) \) to be a function such that \( T'_{\alpha}(s) = \kappa_{\alpha}(s)N_{\alpha}(s) \). (Note that unlike in two dimensions, defining things this way makes the curvature always non-negative.)

Question 7.33. Consider the parameterized helicoid

\[
\alpha(t) = (a \cos t, a \sin t, bt).
\]

(1) Find a unit speed reparameterization.
(2) Calculate \( T_{\alpha}(s) \), \( \kappa_{\alpha}(s) \) and \( N_{\alpha}(s) \).

It is always convenient to refer to vectors in relation to an orthonormal basis. Such a basis consists of unit vectors that are pairwise orthogonal. To complete \( T_{\alpha}(s) \) and \( N_{\alpha}(s) \) to such a basis for \( \mathbb{R}^3 \), we’ll need a third vector orthogonal to both. Such a vector is easily found by the cross product. We define the binormal to be the vector \( B_{\alpha}(s) = T_{\alpha}(s) \times N_{\alpha}(s) \).

Question 7.34. Calculate \( B_{\alpha}(s) \) for the helicoid.

In 2D we took the derivative of \( N_{\alpha}(s) \), and discovered the result was equal to \(-\kappa(s)T_{\alpha}(s)\). Unfortunately, in 3D there is no such nice relationship between \( N'_{\alpha}(s) \) and \( T_{\alpha}(s) \). The most we can say is that
$N'_\alpha(s)$ must be orthogonal to $N_\alpha(s)$, for the same reason illustrated above for why $T'_\alpha(s)$ is orthogonal to $T_\alpha(s)$.

Since $N'_\alpha(s)$ is orthogonal to $N_\alpha(s)$, it must be in the plane spanned by $T_\alpha(s)$ and $B_\alpha(s)$. Thus, we should be able to express $N'_\alpha(s)$ as a linear combination of these two vectors. The coefficient of $T_\alpha(s)$ in such an expression is found by taking the dot product of $N'_\alpha(s)$ with $T_\alpha(s)$. We’ll find the value of this dot product by differentiating both sides of the equation $T_\alpha(s) \cdot N_\alpha(s) = 0$ to obtain

$$
0 = T'_\alpha(s) \cdot N_\alpha(s) + T_\alpha(s) \cdot N'_\alpha(s) \\
= \kappa_\alpha(s)N_\alpha(s) \cdot N_\alpha(s) + T_\alpha(s) \cdot N'_\alpha(s) \\
= \kappa_\alpha(s) + T_\alpha(s) \cdot N'_\alpha(s)
$$

Thus, $T_\alpha(s) \cdot N'_\alpha(s) = -\kappa_\alpha(s)$. We conclude that we can express $N'_\alpha(s)$ as a linear combination of $T_\alpha(s)$ and $B_\alpha(s)$ by

$$
N'_\alpha(s) = -\kappa_\alpha(s)T_\alpha(s) + \tau_\alpha(s)B_\alpha(s),
$$

for some function $\tau_\alpha(s)$. The function $\tau_\alpha(s)$ that makes this equation true is called the torsion of $\alpha$.

Our final task is to express the vector $B'_\alpha(s)$ in the orthonormal basis $\{T_\alpha(s), N_\alpha(s), B_\alpha(s)\}$. To do this, we begin with the defining equation $B_\alpha(s) = T_\alpha(s) \times N_\alpha(s)$ and differentiate both sides:

$$
B'_\alpha(s) = T'_\alpha(s) \times N_\alpha(s) + T_\alpha(s) \times N'_\alpha(s) \\
= \kappa_\alpha(s)N_\alpha(s) \times N_\alpha(s) + T_\alpha(s) \times N'_\alpha(s) \\
= 0 + T_\alpha(s) \times (-\kappa_\alpha(s)T_\alpha(s) + \tau_\alpha(s)B_\alpha(s)) \\
= -T_\alpha(s) \times \kappa_\alpha(s)T_\alpha(s) + T_\alpha(s) \times \tau_\alpha(s)B_\alpha(s) \\
= 0 + \tau_\alpha(s)T_\alpha(s) \times B_\alpha(s) \\
= -\tau_\alpha(s)N_\alpha(s)
$$

This final equation probably gives the easiest way to calculate torsion:

$$
\tau_\alpha(s) = -B'_\alpha(s) \cdot N_\alpha(s).
$$

**Question 7.35.** Calculate $\tau_\alpha(s)$ for the helicoid parameterized above.

The three fundamental equations we have found in this section are called the Frenet-Serret equations. They are:
\[
\begin{align*}
T'_{\alpha}(s) &= \kappa_{\alpha}(s)N_{\alpha}(s) \\
N'_{\alpha}(s) &= -\kappa_{\alpha}(s)T_{\alpha}(s) + \tau_{\alpha}(s)B_{\alpha}(s) \\
B'_{\alpha}(s) &= -\tau_{\alpha}(s)N_{\alpha}(s)
\end{align*}
\]

It is important to keep track of what here is a definition and what is a theorem:

1. The first equation makes sense because you can prove that \(T'_{\alpha}(s)\) is orthogonal to \(T_{\alpha}(s)\). The function \(\kappa_{\alpha}(s)\) and vector \(N_{\alpha}(s)\) are defined so that the equation is true.

2. The second equation makes sense because you can prove that \(N'_{\alpha}(s)\) is some linear combination of \(T_{\alpha}(s)\) and \(B_{\alpha}(s)\). You can also prove that the coefficient of \(T_{\alpha}(s)\) in such an expression must be \(-\kappa_{\alpha}(s)\). The function \(\tau_{\alpha}(s)\) is then defined so that the equation is true.

3. The third equation can be proved.

We illustrate the power of these ideas with the following theorem:

**Theorem 7.36.** A unit speed curve \(\alpha : [a, b] \to \mathbb{R}^3\) with nowhere zero curvature lies in a plane if and only if \(\tau_{\alpha}(s) = 0\).

**Proof.** First, suppose \(\alpha(s)\) is contained in a plane, \(P\). Let \(V\) be a unit normal vector to \(P\). Then \((\alpha(s) - \alpha(0)) \cdot V = 0\). Differentiating, this gives us \(\alpha'(s) \cdot V = 0\). Now recall that \(T_{\alpha}(V)\) points in the same direction as \(\alpha'(s)\). We conclude \(T_{\alpha}(s)\) is always orthogonal to \(V\).

We now differentiate the equation \(T_{\alpha}(s) \cdot V = 0\) to obtain
\[
0 = T'_{\alpha}(s) \cdot V = \kappa_{\alpha}(s)N_{\alpha}(s) \cdot V
\]

As \(\kappa_{\alpha}(s)\) is never zero, we conclude \(N_{\alpha}(s)\) is orthogonal to \(V\). Since both \(T_{\alpha}(s)\) and \(N_{\alpha}(s)\) are orthogonal to \(V\), it follows that \(B_{\alpha}(s) = T_{\alpha}(s) \times N_{\alpha}(s)\) is parallel to \(V\). Since they are both unit vectors, then we have \(B_{\alpha}(s) = V\). Thus, \(B'_{\alpha}(s) = 0\). However, by the third Frenet-Serret equation, \(B'_{\alpha}(s) = -\tau_{\alpha}(s)N_{\alpha}(s)\). We conclude \(\tau_{\alpha}(s) = 0\).

Now we assume \(\tau_{\alpha}(s) = 0\), and show that \(\alpha(s)\) must lie in a plane. The proof is essentially the argument given above, in reverse order. We begin with the third Frenet-Serret equation:
\[
B'_{\alpha}(s) = -\tau_{\alpha}(s)N_{\alpha}(s) = 0
\]
Thus, $B_\alpha(s)$ must be a constant vector. As $T_\alpha(s)$ and $B_\alpha(s)$ are always orthogonal, it follows that $T_\alpha(s) \cdot B_\alpha(s) = 0$. Furthermore, $T_\alpha(s)$ and $\alpha'(s)$ point in the same direction, so we have $\alpha'(s) \cdot B_\alpha(s) = 0$. As $B_\alpha(s)$ is constant, the antiderivative of this equation is

$$(\alpha(s) - \alpha(0)) \cdot B_\alpha(s) = 0,$$

and thus $\alpha(s)$ lies in the plane orthogonal to $B_\alpha(s)$. $\square$

---

**Homework**

**Problem 23.** Show that if $\kappa_\alpha(s) = 0$ and $\tau_\alpha(s) = 0$ then $\alpha$ is a line.

**Problem 24.** Show that if $\kappa_\alpha(s) = c \neq 0$ and $\tau_\alpha(s) = 0$ then $\alpha$ is a circle of radius $\frac{1}{c}$. 
LESSON 8
Computations with non-unit speed curves.

As with planar curves, it will be useful to have a formula to calculate curvature without having a unit-speed parameterization. To motivate this, we examine the formula we got in the planar case:

\[ \kappa_\alpha(s) = \frac{\alpha''(t) \cdot R_{90}(\alpha'(t))}{|\alpha'(t)|^3} \]

One way in which curvature in three dimensions differs from curvature in two dimensions is that it is always positive. Hence, an initial guess at the correct formula might be to take the absolute value of the above expression.

Rotating 90 degrees in 3 dimensions is also problematic, because this requires a preferred plane in which to rotate. To address this, we will reconceive of the numerator as follows. Let \( V = \langle a, b \rangle \) and \( W = \langle c, d \rangle \). Then

\[ V \cdot R_{90}(W) = \langle a, b \rangle \cdot \langle -d, c \rangle = bc - ad \]

Now we think of \( V \) and \( W \) are three dimensional vectors in the \( xy \)-plane, i.e. \( V = \langle a, b, 0 \rangle \) and \( W = \langle c, d, 0 \rangle \). Then notice that

\[ W \times V = \begin{vmatrix} i & j & k \\ c & d & 0 \\ a & b & 0 \end{vmatrix} = (bc - ad)k \]

Thus, \( |V \cdot R_{90}(W)| = |W \times V| \), where on the left we think of \( V \) and \( W \) are 2-dimensional vectors, and on the right as three.

Another way to think about this equality is as follows. Suppose \( \theta \) is the angle between \( V \) and \( W \). Then

\[ |V \cdot R_{90}(W)| = |V||W| \cos(\theta + 90) = |V||W| \sin \theta = |W \times V|. \]

Using this equivalence, we might guess that the correct formula for curvature in three dimensions is

\[ \kappa_\alpha(s) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}. \]

We prove this guess is correct now.

**Theorem 8.37.** Let \( \alpha(t) \) be a regular curve in \( \mathbb{R}^3 \) with non-zero curvature. Then

\[ \kappa_\alpha(s) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \]
Proof. Recall that the arc length parameter $s$ is the antiderivative of $|\alpha'(t)|$, and thus $\frac{ds}{dt} = |\alpha'(t)|$. Hence, $\alpha'(t) = \frac{d\alpha}{ds} = T_a(s)|\alpha'(t)|$.

Next, observe

\[
\alpha'(t) \times \alpha''(t) = T_a(s)|\alpha'(t)| \times (T_a(s)|\alpha'(t)|)' \\
= T_a(s)|\alpha'(t)| \times \left( T_a(s)|\alpha'(t)|' + T'_a(s)|\alpha'(t)| \right) \\
= T_a(s)|\alpha'(t)| \times T'_a(s)|\alpha'(t)|' + T_a(s)|\alpha'(t)| \times T'_a(s)|\alpha'(t)|^2 \\
= 0 + T_a(s)|\alpha'(t)| \times \kappa_a(s)N_a(s)|\alpha'(t)|^2 \\
= \kappa_a(s)|\alpha'(t)|^3 T_a(s) \times N_a(s)
\]

and thus,

\[
|\alpha'(t) \times \alpha''(t)| = \kappa_a(s)|\alpha'(t)|^3
\]

and the desired result follows. \( \square \)

**Question 8.38.** Use the formula above to re-derive the curvature of the helicoid

\[(a \cos t, a \sin t, bt)\]

**Homework**

**Problem 25.** Compute the curvature of the *twisted cubic*

\[(t, t^2, t^3)\]

**Problem 26.** Suppose $\alpha : [a, b] \to \mathbb{R}^3$ is a regular unit speed curve. Then $\gamma(s) = T_a(s)$ and $\delta(s) = B_a(s)$ can be thought of as curves on the unit sphere.

1. Show that the total curvature of $\alpha$ is the length of $\gamma$:

\[
\text{length}(\gamma) = \int_a^b \kappa_a(s) \, ds.
\]

2. Show that $\gamma$ is regular when $\kappa_a > 0$ and

\[
\kappa_\gamma(s) = \sqrt{1 + \left( \frac{\tau_a(s)}{\kappa_a(s)} \right)^2}.
\]

3. Show that $\delta$ is regular when $|\tau_a(s)| > 0$ and

\[
\kappa_\delta(s) = \sqrt{1 + \left( \frac{\kappa_a(s)}{|\tau_a(s)|} \right)^2}.
\]
LESSON 9

Space curves are determined by their curvature and torsion.

**Theorem 9.39.** Suppose $\alpha(s)$ and $\beta(s)$ are two curves with curvature function $\kappa(s)$ and torsion function $\tau(s)$. Then there is a rigid motion $M$ such that $M\alpha = \beta$.

**Proof.** Let $M$ be the unique rigid motion that takes $\alpha(0)$, $T_\alpha(0)$, $N_\alpha(0)$ and $B_\alpha(0)$ to $\beta(0)$, $T_\beta(0)$, $N_\beta(0)$ and $B_\beta(0)$. We claim that $M\alpha = \beta$.

Let $f(s)$ be defined as follows:

$$f(s) = T_{M\alpha}(s) \cdot T_\beta(s) + N_{M\alpha}(s) \cdot N_\beta(s) + B_{M\alpha}(s) \cdot B_\beta(s).$$

Then note that

$$f(0) = T_{M\alpha}(0) \cdot T_\beta(0) + N_{M\alpha}(0) \cdot N_\beta(0) + B_{M\alpha}(0) \cdot B_\beta(0) - \kappa(s)(N_{M\alpha}(s) \cdot T_\beta(s) + T_{M\alpha}(s) \cdot N_\beta(s)) - \tau(s)(N_{M\alpha}(s) \cdot B_\beta(s) + B_{M\alpha}(s) \cdot N_\beta(s)) = 3$$

We now calculate the derivative of $f(s)$:

$$f'(s) = (T'_{M\alpha}(s) \cdot T_\beta(s) + T_{M\alpha}(s) \cdot T'_\beta(s))$$
$$+ (N'_{M\alpha}(s) \cdot N_\beta(s) + N_{M\alpha}(s) \cdot N'_\beta(s))$$
$$+ (B'_{M\alpha}(s) \cdot B_\beta(s) + B_{M\alpha}(s) \cdot B'_\beta(s))$$
$$= \kappa(s)(N_{M\alpha}(s) \cdot T_\beta(s) + T_{M\alpha}(s) \cdot N_\beta(s))$$
$$- \kappa(s)(T_{M\alpha}(s) \cdot N_\beta(s) + N_{M\alpha}(s) \cdot T_\beta(s))$$
$$+ \tau(s)(B_{M\alpha}(s) \cdot N_\beta(s) + N_{M\alpha}(s) \cdot B_\beta(s))$$
$$- \tau(s)(N_{M\alpha}(s) \cdot B_\beta(s) + B_{M\alpha}(s) \cdot N_\beta(s))$$
$$= 0$$

Hence, $f(s)$ must be a constant function. Since $f(0) = 3$, we conclude $f(s) = 3$ for all $s$. Now note that each of the summands in the definition of $f(s)$ is the dot product of two unit vectors, and hence must be at most one. Since the sum is always 3, we conclude each summand
is exactly one. Thus,

\[ T_{M\alpha}(s) = T_\beta(s) \]
\[ N_{M\alpha}(s) = N_\beta(s) \]
\[ B_{M\alpha}(s) = B_\beta(s) \]

The first of these equations is the only one we are actually interested in. Since \( T_{M\alpha}(s) \) is the derivative of \( M\alpha(s) \) and \( T_\beta(s) \) is the derivative of \( \beta(s) \), the two curves differ by at most a constant. However, since \( M\alpha(0) = \beta(0) \), we conclude the curves must actually be the same. \( \square \)
Theorem 10.40 (Fenchel, 1929). The total curvature of a closed curve \( \alpha : [a,b] \to \mathbb{R}^3 \) is greater than or equal to 2\( \pi \), i.e.
\[
\int_a^b \kappa_\alpha(s) \, ds \geq 2\pi.
\]

The following proof is essentially due to R. Horn in 1971. It relies on two very believable facts, which we will prove later in the course:

1. The shortest path between two antipodal points on a unit sphere has length \( \pi \).
2. If two points on a unit sphere are closer than \( \pi \), then the shortest path between them is unique, and is realized by a subarc of a great circle (i.e. a minor arc).

Proof. In a previous homework problem, you showed that the total curvature of \( \alpha(s) \) is equal to the length of the spherical curve \( \beta(s) = T_\alpha(s) \). Hence, Fenchel’s Theorem asserts that if \( \alpha \) is closed, then the length of \( \beta \) is greater than or equal to 2\( \pi \). By way of contradiction, we assume the length of \( \beta \) is less than 2\( \pi \).

Let \( p \) and \( q \) be any two points on \( \beta \) that break it up into two pieces \( \beta_1 \) and \( \beta_2 \) of equal length, therefore both less than \( \pi \). Then the distance from \( p \) to \( q \) along the sphere is less than \( \pi \) so there is a unique minor arc from \( p \) to \( q \). Let \( N \) be the midpoint of this arc.

Consider the function \( f(s) = \alpha(s) \cdot N. \) Since \( \alpha \) is closed, \( f(a) = f(b) \), and hence \( f \) attains a maximum and minimum on the interval \([a,b] \). At these values of \( s \) we have
\[
0 = f'(s) = \alpha'(s) \cdot N = T_\alpha(s) \cdot N = \beta(s) \cdot N,
\]
so there are at least two points on the curve \( \beta \) on the equator defined by the pole \( N \). Let \( r \) be one such point. Without loss of generality, assume \( r \in \beta_1 \).

Let \( \beta_1' \) be the curve obtained from \( \beta_1 \) by rotating it one-half turn about the axis through \( N \), so that \( p \) goes to \( q \) and \( q \) to \( p \) while \( r \) goes to the antipodal point \( r' \). The curve formed by \( \beta_1 \) and \( \beta_1' \) has the same length as the original curve \( \beta \), but it contains a pair of antipodal points so it must have length at least 2\( \pi \). This contradicts our hypothesis that the length of \( \beta \) was less than 2\( \pi \). \( \square \)
Here is an interesting generalization of Fenchel’s Theorem. Unfortunately, it’s proof is beyond the scope of this course.

**Theorem 10.41** (Fary 1949, Milnor 1950). *The total curvature of a knotted simple closed curve is at least $4\pi$.\*

*(To say a curve is *knotted* means that it is not deformable through simple closed curves to a round circle.)*

---

**Homework**

**Problem 27.** Show that if equality holds in Fenchel’s Theorem then $\alpha(s)$ is a (convex*\*) planar curve. (*You don’t have to show convexity, but its a nice challenge!*)
LESSON 11
Surfaces

Definition 11.42. A parameterization of a set \( M \subset \mathbb{R}^3 \) is a \((C^3)\) one-to-one function \( S : U \rightarrow M \), where \( U \) is an open subset of \( \mathbb{R}^2 \).

Recall that the arclength of a parameterized curve \( \alpha(t) \) is the integral of its speed, and that a curve is defined to be regular if its speed is non-zero. We wish to do the same thing for surfaces: find a function whose integral gives surface area, and then define a “regular surface” to be one for which that function is non-zero.

Choose a grid of evenly spaced points \( \{(x_i, y_j)\} \) in \( U \). Let \( \Delta x = x_{i+1} - x_i \) and \( \Delta y = y_{j+1} - y_j \). The points \( \{S(x_i, y_j)\} \) form a grid in \( M \), and the area of one cell of this grid is:

\[
\left| \left( S(x_{i+1}, y_j) - S(x_i, y_j) \right) \times \left( S(x_i, y_{j+1}) - S(x_i, y_j) \right) \right| \Delta x \Delta y
\]

\[
\approx \left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right| \Delta x \Delta y
\]

Thus, the total surface area becomes a Riemann sum which converges to

\[
\text{Area}(M) = \iint_U \left| \frac{\partial S}{\partial x} \times \frac{\partial S}{\partial y} \right| \, dx \, dy.
\]

We define a surface to be regular precisely when the integrand in this expression is non-zero. For the sake of simplicity, we will write \( S_x \) and \( S_y \) for the two partial derivatives of \( S \).

Definition 11.43. A parameterization \( S : U \subset \mathbb{R}^2 \rightarrow M \subset \mathbb{R}^3 \) is regular when the vector \( S_x \times S_y \) is non-zero at every point of \( U \).

Note that for fixed \( y_0 \), \( S(x, y_0) \) is a parameterized space curve, and \( S_x \) is just the tangent vector to this curve. The condition that \( S_x \times S_y \) is non-zero is equivalent to saying that these two tangent vectors are linearly independent.

As both \( S_x \) and \( S_y \) are tangent to \( M \), their span is a tangent plane to \( M \). Furthermore, since the cross product of two vectors is orthogonal to both, the vector \( S_x \times S_y \) is orthogonal to \( M \). Thus, for each \( p \in U \), a unit normal vector to \( M \) at the point \( S(p) \) is given by

\[
N_S(p) = \frac{S_x(p) \times S_y(p)}{|S_x(p) \times S_y(p)|}.
\]
Definition 11.44. A regular surface is a subset $S$ of $\mathbb{R}^3$ such that for each $p \in S$ there is a regular parameterization of a subset of $S$ that contains $p$.

Here are a few examples of common regular surfaces:

1. The graph of any differentiable real-valued function $f(x, y)$.
2. A sphere
3. A cylinder

Homework

Problem 28. Let $\alpha(s) = (0, f(s), g(s))$ be a regular parameterized curve. Then the surface of revolution obtained by rotating $\alpha$ about the $z$-axis is given by

$$S(s, t) = (f(s) \cos t, f(s) \sin t, g(s)), \quad 0 \leq t < 2\pi.$$  

(1) Show that the cone, sphere, and cylinder are surfaces of revolution.

(2) Give a formula for $N_S(s, t)$ in terms of $f(t)$ and $g(t)$.

Problem 29. Let $\alpha, \beta : [a, b] \to \mathbb{R}^3$ be a regular parameterized curves, where $\beta$ is never at the origin. The parameterized surface

$$S(s, t) = \alpha(s) + t\beta(s), \quad s \in [a, b], \quad t \in \mathbb{R}$$

is called a ruled surface with directrix $\alpha$. Note that for such a $S(s, t)$, if you fix $s$ and vary $t$ you get a line on $S(s, t)$. These lines fill the surface and are called the rulings.

(1) Show that the cone and cylinder are ruled surfaces. In each case describe the directrix and rulings.

(2) The helicoid (i.e. the graph of $z = \theta$ in cylindrical coordinates) is a ruled surface. Describe the directrix and rulings.

(3) Give a formula for $N_S(s, t)$ in terms of $\alpha(t)$ and $\beta(t)$.
**LESSON 12**

**Covariant Differentiation**

Before proceeding with our study of surfaces, we must generalize an operation from Calc III. Recall that, given a function $f(x,y)$, a point $p \in \mathbb{R}^2$, and a unit vector $V$ based at $p$, we can compute the slope of the tangent line to the graph of $z = f(x,y)$, at $p$, in the $V$ direction as follows:

$$\nabla_V f(p) = \nabla f(p) \cdot V = \left\langle \frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p) \right\rangle \cdot V$$

When $V$ is not a unit vector, then above quantity represents the rate of change of the function $f$ at $p$, when traveling in the direction of $V$ with speed $|V|$. In other words, if $\alpha(t)$ is a parameterized curve in $\mathbb{R}^2$ such that $\alpha(0) = p$ and $\alpha'(0) = V$, then

$$\frac{d}{dt} f(\alpha(t))\bigg|_{t=0} = \nabla f(\alpha(0)) \cdot \alpha'(0) = \nabla f(p) \cdot V = \nabla_V f(p).$$

(Note how the chain rule works for composition of multivariable functions with functions of one variable!)

In Calc III you may have called $\nabla_V f(p)$ the *directional derivative*. In this class we’ll see this is a special case of a general operation called *covariant differentiation*.

Now suppose $S(x,y) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parameterization of a surface. Then $S(x,y) = (f(x,y), g(x,y), h(x,y))$. If $p$ is again a point of $\mathbb{R}^2$ and $V$ is a vector in $\mathbb{R}^2$ based at $p$, we can now define

$$\nabla_V S(p) = \langle \nabla_V f(p), \nabla_V g(p), \nabla_V h(p) \rangle.$$

Geometrically, this defines a tangent vector to $S$ at the point $S(p)$. To see *which* tangent vector, suppose $V = \langle a, b \rangle$. Then

$$\nabla_V S(p) = \langle \nabla_V f(p), \nabla_V g(p), \nabla_V h(p) \rangle$$
$$= \langle \nabla f(p) \cdot V, \nabla g(p) \cdot V, \nabla h(p) \cdot V \rangle$$
$$= \langle af_x(p) + bg_x(p), ag_x(p) + bh_x(p), ah_x(p) + bh_x(p) \rangle$$
$$= a(f_x(p), g_x(p), h_x(p)) + b(f_y(p), g_y(p), h_y(p))$$
$$= aS_x(p) + bS_y(p)$$

In other words, when $V = \langle a, b \rangle$ the vector $\nabla_V S$ is the vector $V = \langle a, b \rangle$ in the $\{S_x, S_y\}$ basis!

Another way to view $\nabla_V S(p)$ is as follows. Suppose $\alpha(t)$ is a parameterized curve in $\mathbb{R}^3$ such that $\alpha(0) = p$ and $\alpha'(0) = V$. Then $S(\alpha(t))$
is a parameterized curve in $\mathbb{R}^3$. The vector $\nabla_V S(p)$ is then the tangent vector to this curve at $S_V(p)$. That is,

$$\nabla_V S(p) = \left. \frac{d}{dt} S(\alpha(t)) \right|_{t=0}$$

**Question 12.45.** Consider the parameterization of the sphere:

$$S(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$$

1. Calculate the covariant derivative of $S$, at the the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$, in the direction $\left<\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right>$.

2. Show that this vector is tangent to the sphere by showing its dot product with $S\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is zero.

We can use the above definition to define the derivative of a vector field on the surface $S$ as follows. Suppose that for each $p \in \mathbb{R}^2$, $W(p)$ represents a vector in $\mathbb{R}^3$ based at the point $S(p)$. Again, let $V$ be a vector in $\mathbb{R}^2$ based at $p$. Then we can think of $W$ as itself being another parameterized surface in $\mathbb{R}^3$, and define $\nabla_V W(p)$ just as above. Note that the derivative of a vector field on a surface will then be another vector field defined on the surface. (This is just like the situation for curves. Recall that for a parameterized curve $\alpha(s)$, the derivative of $T_\alpha(s)$ is $\kappa_\alpha(s) N_\alpha(s)$, another vector field on $\alpha$.)

**Question 12.46.** Calculate the vector field $S_\theta = \frac{\partial S}{\partial \theta}$ for the parameterization of the sphere given above. Then calculate the covariant derivative of this vector field in the $\langle 1, 0 \rangle$ direction.

---

**Homework**

**Problem 30.** Show that the covariant derivative of the sphere, in any direction, is tangent to the sphere.
LEsson 13
Sectional Curvature

Suppose $S : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is a parameterized surface. Let $N_S(p)$ denote the normal vector to $S$ at $S(p)$, and suppose $\mathcal{V}$ is a unit tangent vector to $S$ at the same point. Then $N_S(p)$ and $\mathcal{V}$ together define a plane $\Pi$ in $\mathbb{R}^3$. Let $\alpha(s)$ be a parameterization of the curve in $U \subset \mathbb{R}^2$ so that $S(\alpha(s))$ (which we will denote $S_{\alpha}(s)$) is a unit speed parameterization of the curve $S \cap \Pi$, $\alpha(0) = p$ and $\frac{d}{ds} S(\alpha(s)) \bigg|_{s=0} = T_{S_{\alpha}(0)} = \mathcal{V}$. Let $V = \alpha'(0)$. Thus,

$$V = \left. \frac{d}{ds} S(\alpha(s)) \right|_{s=0} = \nabla_V S(p)$$

Since $S_{\alpha}(s)$ lies in the plane $\Pi$, we can use all of the machinery of planar curves to study it. In particular, we will be interested in the curvature of $S_{\alpha}(s)$ when $s = 0$. This number is called the sectional curvature of $S$ at $p$ in the $\mathcal{V}$ direction.

Recall that for planar curves, $T'_{S\alpha}(s) = \kappa_{S\alpha}(s) N_{S\alpha}(s)$. Thus, $\kappa_{S\alpha}(s) = T'_{S\alpha}(s) \cdot N_{S\alpha}(s)$. At $s = 0$, we thus have

$$\kappa_{S\alpha}(0) = T'_{S\alpha}(0) \cdot N_{S\alpha}(0) = T'_{S\alpha}(0) \cdot N_{S}(\alpha(0))$$

(CAUTION! Note that in general, $N_{S\alpha}(s) \neq N_S(\alpha(s))$. The first represents the normal vector to the curve $S_{\alpha}$, which will lie in the plane $\Pi$. The second is the normal vector to $S$ at the point $\alpha(s)$, which is only guaranteed to lie in the plane $\Pi$ when $s = 0$.)

Since $T_{S\alpha}(s)$ is always tangent to $S$, it follows that for all $s$,

$$0 = T_{S\alpha}(s) \cdot N_S(\alpha(s))$$

Differentiating both sides of the last equation gives us

$$0 = T'_{S\alpha}(s) \cdot N_S(\alpha(s)) + T_{S\alpha}(s) \cdot \left. \frac{d}{ds} N_S(\alpha(s)) \right|_{s=0}$$

Thus, at $s = 0$ we have

$$\kappa_{S\alpha}(0) = T'_{S\alpha}(0) \cdot N_S(\alpha(0))$$

$$= T_{S\alpha}(0) \cdot \left. \frac{d}{ds} N_S(\alpha(s)) \right|_{s=0}$$

$$= -V \cdot \nabla_{\mathcal{V}} N_S(p)$$

Note the analogy with planar curves, where

$$\kappa_{S\alpha}(s) = -T_{S\alpha}(s) \cdot N'_{\alpha}(s).$$
In both cases, curvature is the dot product of the tangent vector with the derivative of the normal vector.

**Question 13.47.** Let $S$ be the graph of $z = xy$. Compute the sectional curvature of $S$ at $(2, 2, 4)$ in the direction $\left< \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right>$ in two ways:

1. Find the curve $\alpha(t)$ above and directly compute the curvature of the planar curve $S(\alpha(t))$ at $t = 0$.
2. Use the above formula.
LESSON 14
The Second Fundamental Form

In the previous section we saw that the sectional curvature of a surface \( S \), in the direction \( V \) of a tangent vector to \( S \), is given by

\[
\kappa_S(V) = -V \cdot \nabla_V N(p).
\]

Here \( V = aS_x + bS_y \) is a vector in \( \mathbb{R}^3 \) and \( V = \langle a, b \rangle \) is a vector in \( \mathbb{R}^2 \).

We now do a computation:

\[
\kappa_S(V) = -V \cdot \nabla_V N = -(aS_x + bS_y) \cdot \nabla_{\langle a, b \rangle} N = -(aS_x + bS_y) \cdot (aN_x + bN_y) = -a^2S_x \cdot N_x - abS_x \cdot N_y - abS_y \cdot N_x - b^2S_y \cdot N_y
\]

Now notice that \( S_x \) and \( N \) are orthogonal, so that \( S_x \cdot N = 0 \). Taking the derivative of this equation with respect to \( x \) yields \( S_{xx} \cdot N + S_x \cdot N_x = 0 \), and thus \( S_{xx} \cdot N = -S_x \cdot N_x \). Similarly, \( S_{xy} \cdot N = -S_x \cdot N_y \) and \( S_{yy} \cdot N = -S_y \cdot N_y \). Thus, we can rewrite the above as

\[
\kappa_S(V) = -a^2S_x \cdot N_x - abS_x \cdot N_y - abS_y \cdot N_x - b^2S_y \cdot N_y = a^2S_{xx} \cdot N + abS_{xy} \cdot N + abS_{yx} \cdot N + b^2S_{yy} \cdot N = \langle a, b \rangle \cdot \begin{bmatrix} N_x \cdot S_{xx} & N_x \cdot S_{xy} \\ N_x \cdot S_{yx} & N_x \cdot S_{yy} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
\]

The matrix that appears here is called the \textit{second fundamental form}, and is denoted II. Hence, we can write compactly that the sectional curvature in the \( V = aS_x + bS_y \) direction is:

\[
\kappa_S(V) = \langle a, b \rangle \cdot II(\langle a, b \rangle)
\]
LESSON 15
The Shape Operator

Earlier we saw that the sectional curvature of a surface $S$, in the direction $V = aS_x + bS_y$ of a tangent vector to $S$, is given by

$$\kappa_S(V) = -V \cdot \nabla_{\langle a,b \rangle} N.$$ 

**Definition 15.48.** Suppose $V = aS_x + bS_y$ is a unit tangent vector to $S$. Then the shape operator of $S$ in the $V$ direction is

$$S(V) = -\nabla_{\langle a,b \rangle} N.$$ 

Note that with this definition, we can say that the sectional curvature of $S$ in the $V$ direction is given simply by

$$\kappa_S(V) = V \cdot S(V)$$

**Theorem 15.49.** Suppose $V = aS_x + bS_y$ is a unit tangent vector to $S$ at $p$. Then $S(V)$ is a vector tangent to $S$.

*Proof.* For all $p \in S$, $N \cdot N = 1$. Taking the covariant derivative of both sides in the $\langle a, b \rangle$ direction gives us $N \cdot \nabla_{\langle a, b \rangle} N = 0$. We conclude $S(V) = -\nabla_{\langle a, b \rangle} N$ is orthogonal to $N$, and thus tangent to $S$.  

It follows that $S$ is an operator that takes tangent vectors to $S$ and returns tangent vectors.

**Definition 15.50.** The set of tangent vectors to $S$ at $p$ forms a tangent plane, denote $T_p S$.

Using this notation, the previous theorem says $S : T_p S \to T_p S$.

**Theorem 15.51.** The shape operator, when viewed as a map from $T_p S$ to $T_p S$, is linear.

*Proof.* Suppose $p = S(x, y)$, and $N(p) = (f(x, y), g(x, y), h(x, y))$. Then

$$S(aS_x + bS_y) = -\langle \nabla f \cdot \langle a, b \rangle, \nabla g \cdot \langle a, b \rangle, \nabla h \cdot \langle a, b \rangle \rangle .$$

As each individual dot product is linear, so is the resulting function on $T_p S$.  

Since the shape operator takes tangent vectors to tangent vectors, it follows that for each $a, b$ there is a $c, d$ so that

$$S(aS_x + bS_y) = cS_x + dS_y$$

The linearity of the shape operator implies there must be a $2 \times 2$ matrix $M$ that relates the pair $a, b$ to the pair $c, d$. We proceed to find this matrix now. First, note that
\[ S(aS_x + bS_y) = -\nabla_{\langle a, b \rangle} N = -aN_x - bN_y \]

So we have
\[ -aN_x - bN_y = cS_x + dS_y \]

Taking the dot product of this equation with both \( S_x \) and \( S_y \) leads to the following pair of equations:
\[ -aN_x \cdot S_x - bN_y \cdot S_x = cS_x \cdot S_x + dS_y \cdot S_x, \]
\[ -aN_x \cdot S_y - bN_y \cdot S_y = cS_x \cdot S_y + dS_y \cdot S_y. \]

In the previous chapter, we noted that 
\[ -N_x \cdot S_x = N \cdot S_{xx}, -N_y \cdot S_x = N \cdot S_{xy}, \quad \text{and} \quad -N_y \cdot S_y = N \cdot S_{yy}. \]

Applying these equalities transforms the above pair of equations to:
\[ aN \cdot S_{xx} + bN \cdot S_{xy} = cS_x \cdot S_x + dS_y \cdot S_x, \]
\[ aN \cdot S_{yx} + bN \cdot S_{yy} = cS_x \cdot S_y + dS_y \cdot S_y. \]

We can write this pair of equations in matrix form as
\[
\begin{bmatrix}
N \cdot S_{xx} & N \cdot S_{xy} \\
N \cdot S_{yx} & N \cdot S_{yy}
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
= \begin{bmatrix}
S_x \cdot S_x & S_y \cdot S_x \\
S_x \cdot S_y & S_y \cdot S_y
\end{bmatrix}
\begin{bmatrix}
c \\
d
\end{bmatrix}
\]

Finally, we conclude that if \( S(aS_x + bS_y) = cS_x + dS_y \) then
\[
\begin{bmatrix}
c \\
d
\end{bmatrix}
= \begin{bmatrix}
S_x \cdot S_x & S_y \cdot S_x \\
S_x \cdot S_y & S_y \cdot S_y
\end{bmatrix}^{-1}
\begin{bmatrix}
N \cdot S_{xx} & N \cdot S_{xy} \\
N \cdot S_{yx} & N \cdot S_{yy}
\end{bmatrix}
\begin{bmatrix}
a \\
b
\end{bmatrix}
\]

The second matrix that appears here is the second fundamental form. The first matrix is called the \textit{first fundamental form}, and is denoted \( I \).

Hence, we can say all of this more compactly as follows:

If \( S(aS_x + bS_y) = cS_x + dS_y \) then \( I^{-1} \Pi(\langle a, b \rangle) = \langle c, d \rangle \).

**Example 15.52.** We compute the shape operator for the sphere of radius 1. This will allow us to calculate all sectional curvatures, which we expect to be 1. We start with the following parameterization:
\[
S(\theta, \phi) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\]

The derivatives of this are:
\[
S_\theta = \langle -\sin \phi \sin \theta, \sin \phi \cos \theta, 0 \rangle
\]
\[
S_\phi = \langle \cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi \rangle
\]

To find the normal vector \( N \) we begin by computing the cross product:
\[
S_\theta \times S_\phi = \langle -\sin^2 \phi \cos \theta, -\sin^2 \phi \sin \theta - \sin \phi \cos \phi \rangle
\]
The magnitude of this vector is \( \sin \phi \), so we must divide by \( \sin \phi \) to get \( N \):

\[
N = \langle -\sin \phi \cos \theta, -\sin \phi \sin \theta - \cos \theta \rangle
\]

Next we compute the second derivatives:

\[
S_{\theta \theta} = \langle -\sin \phi \cos \theta, -\sin \phi \sin \theta, 0 \rangle
\]

\[
S_{\theta \phi} = \langle -\cos \phi \sin \theta, -\cos \phi \cos \theta, 0 \rangle
\]

\[
S_{\phi \phi} = \langle -\sin \phi \cos \theta, -\sin \phi \sin \theta, -\cos \phi \rangle
\]

Now we move on to the dot products:

\[
S_{\theta} \cdot S_{\theta} = \sin^2 \phi
\]

\[
S_{\theta} \cdot S_{\phi} = 0
\]

\[
S_{\phi} \cdot S_{\phi} = 1
\]

\[
N \cdot S_{\theta \theta} = \sin^2 \phi
\]

\[
N \cdot S_{\theta \phi} = 0
\]

\[
N \cdot S_{\phi \phi} = 1
\]

Hence,

\[
S = \begin{bmatrix}
S_{\theta} \cdot S_{\phi} & S_{\phi} \cdot S_{\theta} \\
S_{\theta} \cdot S_{\phi} & S_{\phi} \cdot S_{\phi}
\end{bmatrix}^{-1}
\begin{bmatrix}
N_S \cdot S_{\theta \theta} & N_S \cdot S_{\theta \phi} \\
N_S \cdot S_{\theta \phi} & N_S \cdot S_{\phi \phi}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\sin^2 \phi & 0 \\
0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
\sin^2 \phi & 0 \\
0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

Thus, for any unit tangent vector \( V = aS_{\theta} + bS_{\phi} \) at a point \( p \) of the sphere, \( S(aS_{\theta} + bS_{\phi}) = aS_{\theta} + bS_{\phi} \), i.e. \( S(V) = V \). The sectional curvature in the \( V \) direction is given by

\[
\kappa_S(V) = V \cdot S(V) = V \cdot V = 1
\]

as expected.

**Question 15.53.** Compute the shape operator for the cylinder.

Any \( 2 \times 2 \) matrix \( M \) gives us a way to multiply two vectors to obtain a scalar: if \( V \) and \( W \) represent such vectors, then we can compute \( W \cdot M V \). Note that if \( M \) is the identity matrix, then the result of this operation is just \( W \cdot V \). The dot product has the nice property of being commutative. When \( W \cdot M V = V \cdot M W \), we say \( M \) is a symmetric operator.

The following theorem gives an important property of the shape operator.
**Theorem 15.54.** The shape operator is symmetric.

*Proof.* First we check the following special case:

\[
S_x \cdot S_p(S_y) = -S_x \cdot \nabla_{(0,1)} N \\
= -S_x \cdot (0N_x + N_y) \\
= -S_x \cdot N_y \\
= S_{xy} \cdot N
\]

To see this last equality, differentiate both sides of the equation

\[
S_x \cdot N = 0.
\]

A symmetric computation shows

\[
S_y \cdot S_p(S_x) = S_{yx} \cdot N
\]

Since the mixed partials are equal, we conclude

\[
S_x \cdot S_p(S_y) = S_y \cdot S_p(S_x)
\]

To complete the proof of symmetry in the arbitrary case, just combine this with the linearity of the shape operator:

**Question 15.55.** Show that

\[
(aS_x + bS_y) \cdot S_p(cS_x + dS_y) = (cS_x + dS_y) \cdot S_p(aS_x + bS_y)
\]

\[\square\]

Note that a symmetric operator is represented by a symmetric matrix if all vectors are given with respect to an orthonormal basis. Hence, the matrix representation of the shape operator, found above, would only be symmetric when the vectors \(S_x\) and \(S_y\) are of the same length, and are orthogonal.
Homework

Problem 31. Let $S$ be the graph of $z = f(x, y)$. Show that the matrix of the shape operator for $S$ at the critical points of $f$ is just the Hessian of $f$ (i.e. the matrix of second partials).

Problem 32. Compute the shape operator of the helicoid (i.e. the graph of $z = \theta$ in cylindrical coordinates).

Problem 33. Let $\alpha(s) = (0, f(s), g(s))$ be a regular parameterized curve. Then recall that the surface of revolution obtained by rotating $\alpha$ about the $z$-axis is given by

$$S(s, t) = (f(s) \cos t, f(s) \sin t, g(s)), \quad 0 \leq t < 2\pi.$$ 

Find a formula for the shape operator of a surface of revolution.
LESSON 16
Principal Curvature

For some vectors, \( V = aS_x + bS_y \), we may find that \( S(V) = kV \), for some \( k \). In this case, we would have that \( \text{I}^{-1}\Pi(\langle a, b \rangle) = k\langle a, b \rangle \). In other words, the vector \( \langle a, b \rangle \) is an eigenvector of the matrix \( \text{I}^{-1}\Pi \), and \( k \) is an eigenvalue.

As noted at the end of the previous section, the fact that the shape operator is symmetric means that if we represent the matrix \( \text{I}^{-1}\Pi \) with respect to an orthonormal basis, then it will be symmetric. This observation allows you to prove the following:

**Question 16.56.** Show that the matrix \( \text{I}^{-1}\Pi \) has two real eigenvalues.
*Hint: By choosing an orthonormal basis, you may assume*

\[
\text{I}^{-1}\Pi = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.
\]

Another important fact about the matrix \( \text{I}^{-1}\Pi \) is that its eigenspaces are orthogonal. To see this, suppose \( \mathcal{V}_i \) is a unit eigenvector corresponding the eigenvalue \( k_i \), for \( i = 1, 2 \). Then

\[
\mathcal{V}_1 \cdot S(\mathcal{V}_2) = \mathcal{V}_1 \cdot (k_2\mathcal{V}_2) = k_2(\mathcal{V}_1 \cdot \mathcal{V}_2).
\]

By a symmetric argument,

\[
\mathcal{V}_2 \cdot S(\mathcal{V}_1) = k_1(\mathcal{V}_2 \cdot \mathcal{V}_1).
\]

But by the symmetry of the shape operator, \( \mathcal{V}_1 \cdot S(\mathcal{V}_2) = \mathcal{V}_2 \cdot S(\mathcal{V}_1) \). Hence,

\[
k_2(\mathcal{V}_1 \cdot \mathcal{V}_2) = k_1(\mathcal{V}_2 \cdot \mathcal{V}_1).
\]

This implies that either \( k_1 = k_2 \) or \( \mathcal{V}_1 \cdot \mathcal{V}_2 = 0 \). The former case contradicts the fact that the eigenvalues are distinct. The latter case implies \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are orthogonal, as desired.

**Theorem 16.57.** If \( k \) is an eigenvalue of \( \text{I}^{-1}\Pi \) corresponding to a unit eigenvector \( \langle a, b \rangle \) then \( k \) is the sectional curvature of \( S \) in the \( \mathcal{V} = aS_x + bS_y \) direction.

**Proof.** By assumption, \( \text{I}^{-1}\Pi(\langle a, b \rangle) = k\langle a, b \rangle \). Thus, \( S(aS_x + bS_y) = k(aS_x + bS_y) \), or \( S(\mathcal{V}) = k\mathcal{V} \). The sectional curvature of \( S \) in the \( \mathcal{V} \) direction is

\[
\kappa_S(\mathcal{V}) = \mathcal{V} \cdot S(\mathcal{V}) = \mathcal{V} \cdot (k\mathcal{V}) = k.
\]
Definition 16.58. The eigenvalues of the matrix $I^{-1}II$ are called the principal curvatures, and the corresponding eigenspaces are the principal directions.

Question 16.59. Compute the principal curvatures and directions for a point on the paraboloid $z = r^2$.

The following theorem shows that it is particularly easy to compute the sectional curvature in a direction $V$, if $V$ is at a known angle $\theta$ from one of the principal directions.

Theorem 16.60 (Euler). Suppose $k_1$ and $k_2$ are the principal curvatures, and unit vectors in the corresponding principal directions are $e_1$ and $e_2$. Suppose $V = \cos \theta e_1 + \sin \theta e_2$. Then the sectional curvature in the $V$ direction is

$$k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$ 

Proof. The proof is a simple computation. The sectional curvature in the $V = \cos \theta e_1 + \sin \theta e_2$ direction is given by

$$\kappa_S(V) = V \cdot S(V) = V \cdot S(\cos \theta e_1 + \sin \theta e_2) = V \cdot (\cos \theta S(e_1) + \sin \theta S(e_2)) = V \cdot (k_1 \cos \theta e_1 + k_2 \sin \theta e_2) = (\cos \theta e_1 + \sin \theta e_2) \cdot (k_1 \cos \theta e_1 + k_2 \sin \theta e_2) = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

□

Question 16.61. Continuing with the paraboloid example, find the curvature in a direction that is halfway between the principal directions.

Homework

Problem 34. Use Euler’s Theorem to show that the principal curvatures are the maximum and minimum sectional curvatures at a given point.
LESSON 17

A summary of ways to compute sectional curvature

In this section we review all of the ways we have seen to compute sectional curvature with a specific example. Let $S$ be the saddle surface which is the graph of $z = xy$. Then $S$ is parameterized by

$$S(x, y) = (x, y, xy)$$

We will compute one particular sectional curvature of $S$ at the point $S(2, 1)$.

**Method 1.** Directly from the definition.

Notice that $S(2, y) = (2, y, 2y)$, which is a line. A tangent vector to this line is $\langle 0, 1, 2 \rangle$, which we unitize to $V = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$. At the point $S(2, 1)$ there is some normal vector, $N$ to the surface $S$. The vectors $V$ and $N$ span a plane, which will intersect $S$ in a curve. In this case that curve must be precisely the line $(2, y, 2y)$. The sectional curvature of $S$ at $p$, in the $V$ direction is the (plane) curvature of this line, zero.

**Method 2.** Using the covariant derivative of $N$.

One of the first formulas we saw for the sectional curvature was

$$\kappa_S(V) = -V \cdot \nabla_{(a,b)} N,$$

where $V = aS_x + bS_y$.

To use this, we must find an expression for the normal vector field $N$, defined on $S$. We compute:

$$S_x = \langle 1, 0, y \rangle$$

$$S_y = \langle 0, 1, x \rangle$$

$$S_x \times S_y = \langle -y, -x, 1 \rangle$$

$$N = \frac{S_x \times S_y}{|S_x \times S_y|} = \frac{1}{\sqrt{1 + x^2 + y^2}} \langle -y, -x, 1 \rangle$$

At $p = S(2, 1)$, the vectors $S_x$ and $S_y$ are

$$S_x = \langle 1, 0, 1 \rangle$$

$$S_y = \langle 0, 1, 2 \rangle$$

The vector $V = \frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle$ at the point $p = S(2, 1)$ can thus be written as

$$V = 0S_x + \frac{1}{\sqrt{5}}S_y.$$
Hence, the sectional curvature can be computed as
\[
\kappa_S(V) = -V \cdot \nabla_V N
\]
\[
= -\frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle \cdot \nabla_{\langle 0, \frac{1}{\sqrt{5}} \rangle} N
\]
\[
= -\frac{1}{\sqrt{5}} \langle 0, 1, 2 \rangle \cdot (0N_x + \frac{1}{\sqrt{5}} N_y)
\]
\[
= -\frac{1}{5} \langle 0, 1, 2 \rangle \cdot N_y
\]

To complete this, we compute \( N_y \):
\[
N_y = \frac{1}{(1 + x^2 + y^2)^{\frac{3}{2}}} \langle -1 - x^2, xy, -y \rangle
\]

At the point \( S(2,1) \) this reduces to
\[
N_y = \frac{1}{6^\frac{3}{2}} \langle -5, 2, -1 \rangle
\]

Thus,
\[
\kappa_S(V) = -\frac{1}{5} \langle 0, 1, 2 \rangle \cdot N_y
\]
\[
= -\frac{1}{5} \langle 0, 1, 2 \rangle \cdot \frac{1}{6^\frac{3}{2}} \langle -5, 2, -1 \rangle
\]
\[
= 0
\]

**Method 3. Using the second fundamental form.**

We have also seen that if \( V = aS_x + bS_y \) then
\[
\kappa_S(V) = \langle a, b \rangle \cdot \Pi(\langle a, b \rangle).
\]

To use this, we must compute the second fundamental form, \( \Pi \). For that we’ll need the second partials of the parameterization:
\[
S_{xx} = \langle 0, 0, 0 \rangle
\]
\[
S_{xy} = S_{yx} = \langle 0, 0, 1 \rangle
\]
\[
S_{yy} = \langle 0, 0, 0 \rangle
\]

Thus,
\[
N \cdot S_{xx} = 0
\]
\[
N \cdot S_{xy} = N \cdot S_{yx} = \frac{1}{\sqrt{1 + x^2 + y^2}}
\]
\[
N \cdot S_{yy} = 0
\]
The second fundamental form is then
\[
\II = \begin{bmatrix}
0 & \frac{1}{\sqrt{1+x^2+y^2}} \\
\frac{1}{\sqrt{1+x^2+y^2}} & 0
\end{bmatrix}
\]

At the point \(S(2,1)\) this reduces to
\[
\II = \begin{bmatrix}
0 & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & 0
\end{bmatrix}
\]

The sectional curvature in the \(V = \frac{1}{\sqrt{5}}(0,1,2) = 0S_x + \frac{1}{\sqrt{5}}S_y\) direction is thus
\[
\kappa_S(V) = \left\langle 0, \frac{1}{\sqrt{5}} \right\rangle \cdot \II \left( \left\langle 0, \frac{1}{\sqrt{5}} \right\rangle \right)
\]
\[
= \left\langle 0, \frac{1}{\sqrt{5}} \right\rangle \cdot \begin{bmatrix}
0 & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & 0
\end{bmatrix} \begin{bmatrix}
0 \\
\frac{1}{\sqrt{5}}
\end{bmatrix}
\]
\[
= \left\langle 0, \frac{1}{\sqrt{5}} \right\rangle \cdot \left\langle \frac{1}{\sqrt{30}}, 0 \right\rangle
\]
\[
= 0
\]

Method 4. Using the shape operator.

To use the shape operator, \(S\), we must write down the matrix \(I^{-1}\II\). We already have the second fundamental from, \(\II\). We compute the entries of first fundamental form:

\[
S_x \cdot S_x = 1 + y^2
\]
\[
S_x \cdot S_y = xy
\]
\[
S_y \cdot S_y = 1 + x^2
\]

Thus,
\[
I = \begin{bmatrix}
1 + y^2 & xy \\
x y & 1 + x^2
\end{bmatrix}
\]

The matrix we’ll need to use the shape operator is then
\[
I^{-1}\II = \begin{bmatrix}
1 + y^2 & xy \\
x y & 1 + x^2
\end{bmatrix}^{-1} \begin{bmatrix}
0 & \frac{1}{\sqrt{1+x^2+y^2}} \\
\frac{1}{\sqrt{1+x^2+y^2}} & 0
\end{bmatrix}
\]
At the point $S(2,1)$ this reduces to

$$I^{-1}II = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & 0 \end{bmatrix} = \frac{1}{6\sqrt{6}} \begin{bmatrix} -2 & 5 \\ 2 & -2 \end{bmatrix}$$

Thus, if $\mathcal{V} = \frac{1}{\sqrt{5}}\langle 0, 1, 2 \rangle = 0S_x + \frac{1}{\sqrt{5}}S_y$ we’ll find $\mathcal{S}(\mathcal{V})$ by computing:

$$I^{-1}II \left( \frac{1}{\sqrt{5}} \right) = \frac{1}{6\sqrt{6}} \begin{bmatrix} -2 & 5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \end{bmatrix} = \frac{1}{6\sqrt{30}} \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Thus,

$$\mathcal{S} \left( 0S_x + \frac{1}{\sqrt{5}}S_y \right) = \frac{1}{6\sqrt{30}}(5S_x - 2S_y) = \frac{1}{6\sqrt{30}}(5\langle 1, 0, 1 \rangle - 2\langle 0, 1, 2 \rangle) = \frac{1}{6\sqrt{30}}\langle 5, -2, 1 \rangle$$

The sectional curvature is then

$$\kappa_S(\mathcal{V}) = \mathcal{V} \cdot \mathcal{S}(\mathcal{V}) = \frac{1}{\sqrt{5}}\langle 0, 1, 2 \rangle \cdot \frac{1}{6\sqrt{30}}\langle 5, -2, 1 \rangle = 0$$

**Method 5. Using Euler’s formula.**

To use Euler’s formula, we’ll need the principal curvatures and directions of the surface $S$ at the point $S(2,1)$. These correspond to the eigenvalues and (unit) eigenvectors of the matrix

$$I^{-1}II = \frac{1}{6\sqrt{6}} \begin{bmatrix} -2 & 5 \\ 2 & -2 \end{bmatrix}.$$ 

A little computation gives the eigenvalues as

$$k_\pm = \frac{-2 \pm \sqrt{10}}{6\sqrt{6}}.$$
An eigenvector corresponding to $k_+$ is $\langle 5, \sqrt{10} \rangle$. Thus, the corresponding principal direction is the direction of the vector

$$5S_x + \sqrt{10}S_y = 5\langle 1, 0, 1 \rangle + \sqrt{10}\langle 0, 1, 2 \rangle = \langle 5, \sqrt{10}, 5 + 2\sqrt{10} \rangle$$

The (cosine of the) angle between this principal direction and $V = \frac{1}{\sqrt{5}}\langle 0, 1, 2 \rangle$ is computed with the dot product:

$$\cos \theta = \frac{\langle 5, \sqrt{10}, 5 + 2\sqrt{10} \rangle \cdot \langle 0, 1, 2 \rangle}{||\langle 5, \sqrt{10}, 5 + 2\sqrt{10} \rangle|| ||\langle 0, 1, 2 \rangle||} = \frac{10 + 5\sqrt{10}}{\sqrt{5}\sqrt{100 + 20\sqrt{10}}}$$

Squaring and simplifying yields

$$\cos^2 \theta = \frac{5 + \sqrt{10}}{10}$$

and thus

$$\sin^2 \theta = 1 - \cos^2 \theta = \frac{5 - \sqrt{10}}{10}$$

Euler’s formula now tells us that

$$\kappa_S(V) = k_+ \cos^2 \theta + k_- \sin^2 \theta$$

$$= \left(\frac{-2 + \sqrt{10}}{6\sqrt{6}}\right) \left(\frac{5 + \sqrt{10}}{10}\right) + \left(\frac{-2 - \sqrt{10}}{6\sqrt{6}}\right) \left(\frac{5 - \sqrt{10}}{10}\right)$$

$$= 0$$
LESSON 18
Holonomy and Gaussian Curvature

First, we make the following observations about the entries of the first fundamental form:

\[ \frac{1}{2} G_x = \frac{1}{2} (S_y \cdot S_y)_x = S_y \cdot S_{xy} \]

\[ \frac{1}{2} E_y = \frac{1}{2} (S_x \cdot S_x)_y = S_x \cdot S_{xy} \]

\[ F_x - \frac{1}{2} E_y = (S_x \cdot S_y)_x - S_x \cdot S_{xy} = S_y \cdot S_{xx} \]

By previous results, in a surface with no umbilic points, the principle directions are orthogonal. Thus, there is a pair of vector fields on the surface that are orthogonal. By results from differential equations, this implies that there is a parameterization \( S \) of such a surface for which \( F = S_x \cdot S_y = 0 \) at every point. Such a parameterization is called *orthogonal*. By the above equations, for an orthogonal parameterization we have:

\[ \frac{1}{2} G_x = S_y \cdot S_{xy} \]

\[ \frac{1}{2} E_y = S_x \cdot S_{xy} \]

\[ -\frac{1}{2} E_y = S_y \cdot S_{xx} \]

Every orthogonal parameterization has an orthonormal *frame field*, i.e. an orthonormal basis for every tangent plane, given by:

\[ e_1 = \frac{S_x}{\sqrt{E}}, \quad e_2 = \frac{S_y}{\sqrt{G}} \]

Let \( \alpha(t) \) be a curve in the domain of \( S \) (so that \( S(\alpha(t)) \) is a curve in the surface parameterized by \( S \)). The following definition measures how \( e_1 \) twists toward \( e_2 \) as we move along \( \alpha \):

\[ \phi_{12} = (\nabla_{\alpha'} e_1) \cdot e_2 \]
We now compute:

\[ \phi_{12} = (\nabla_{\alpha'} e_1) \cdot e_2 \]
\[ = \left( \frac{d}{dt} e_1 \right) \cdot e_2 \]
\[ = \left( \frac{d}{dt} \frac{S_x}{\sqrt{E}} \right) \frac{S_y}{\sqrt{G}} \]
\[ = \left( \frac{S'_x}{\sqrt{E}} + \left( \frac{1}{\sqrt{E}} \right)' S_x \right) \frac{S_y}{\sqrt{G}} \]
\[ = \frac{S'_x}{\sqrt{E}} \frac{S_y}{\sqrt{G}} \quad \text{(since } S_x \cdot S_y = 0) \]
\[ = \frac{(S_{xx} x' + S_{xy} y') \cdot S_y}{\sqrt{EG}} \quad \text{(where } \alpha(t) = (x(t), y(t)) \)
\[ = \frac{(S_{xx} \cdot S_y) x' + (S_{xy} \cdot S_y) y'}{\sqrt{EG}} \]
\[ = \frac{-\frac{1}{2} E_y x' + \frac{1}{2} G_x y'}{\sqrt{EG}} \]
\[ = \frac{G_x y' - E_y x'}{2\sqrt{EG}} \]

We now assume \( \alpha \) is a simple closed curve in \( S \), bounding a region \( R \), and integrate the above equality:
\[
\int_{\alpha} \phi_{12}(s) ds = \int_{\partial R} \frac{G_x y' - E_y x'}{2\sqrt{EG}} ds
\]
\[
= \int_{\partial R} \frac{G_x y'}{2\sqrt{EG}} - \frac{E_y x'}{2\sqrt{EG}} ds
\]
\[
= \int_{\partial R} \frac{G_x dy}{2\sqrt{EG}} - \frac{E_y dx}{2\sqrt{EG}}
\]
\[
= \int \int_{R} \left( \frac{G_x}{2\sqrt{EG}} \right)_{x} + \left( \frac{E_y}{2\sqrt{EG}} \right)_{y} \ dx \ dy \quad \text{(Green's Theorem)}
\]
\[
= \int \int_{R} \frac{1}{2\sqrt{EG}} \left( \left( \frac{G_x}{\sqrt{EG}} \right)_{x} + \left( \frac{E_y}{\sqrt{EG}} \right)_{y} \right) \sqrt{EG} \ dx \ dy
\]
\[
= \int \int_{R} \frac{1}{2\sqrt{EG}} \left( \left( \frac{G_x}{\sqrt{EG}} \right)_{x} + \left( \frac{E_y}{\sqrt{EG}} \right)_{y} \right) \ dA
\]
\[
= - \int \int_{R} K \ dA
\]

The last equality is an exercise. The second-to-last equality can be seen because the area element \(dA\) is the area of the parallelogram spanned by \(S_x\) and \(S_y\). As these vectors are orthogonal, this area is just the product of \(|S_x| = \sqrt{E}\) and \(|S_y| = \sqrt{G}\).

Let \(X(t)\) be a unit vector field (tangent to \(S\)) defined on \(S(\alpha(t))\), where \(\alpha : [a,b] \to \mathbb{R}^2\). Then there is a function \(\psi(t)\), defined on \([a,b]\), such that

\[
X(t) = \cos \psi(t) e_1 + \sin \psi(t) e_2
\]

where \(\psi(a) = 0\).

We compute the rate of twisting of \(X\) as we move along \(\alpha\) as follows:

\[
\nabla_{\alpha'} X = \nabla_{\alpha'} (\cos \psi e_1 + \sin \psi e_2)
\]
\[
= \cos \psi \nabla_{\alpha'} e_1 + \sin \psi \nabla_{\alpha'} e_2 + (- \sin \psi e_1 + \cos \psi e_2) \psi'
\]
\[
= \cos \psi \phi_{12} e_2 - \sin \psi \phi_{12} e_1 + (- \sin \psi e_1 + \cos \psi e_2) \psi'
\]
\[
= (\phi_{12} + \psi') (- \sin \psi e_1 + \cos \psi e_2)
\]

If \(X\) does not twist at all as we move along \(\alpha\) (i.e. \(X\) is a parallel vector field), we have \(0 = \nabla_{\alpha'} X\). Thus, we conclude that for a parallel...
vector field, $0 = \phi_{12} + \psi'$, or $\phi_{12} = -\psi'$. Integrating both sides gives us

$$\psi(a) - \psi(b) = \int_{a}^{b} \phi_{12}(t) \, dt$$

When $\alpha$ represents a closed curve, the amount the vector $X(a)$, and the parallel translate, $X(b)$, are rotated with respect to each other is called the holonomy of $\alpha$. Note that holonomy is independent of what vector we used for $X(a)$: the holonomy can always be found by integrating $\phi_{12}$ over $\alpha$.

Combining the results of this section, we have proved the following:

Theorem 18.62. The holonomy around a simple closed curve, $\alpha$, bounding a region $R$ in a surface $S$, is equal to the integral of the Gaussian curvature over $R$. 
Recall that for unit-speed, simple closed plane curves, \( \alpha \), with plane curvature \( \kappa(s) \),

\[
\int_{\alpha} \kappa(s) \, ds = \pm 2\pi
\]

The goal of this section is to generalize this formula to curves on other surfaces. First, we will have to understand the appropriate notion of curvature in this context.

Suppose \( \alpha : [0, L] \to \mathbb{R}^3 \) is a unit-speed space curve. Then a unit tangent vector is \( T(s) = \alpha'(s) \), and the curvature of \( \alpha \) is defined to be the number \( \kappa \) so that \( T'(s) = \kappa N(s) \), where \( |N(s)| = 1 \). If the curve \( \alpha \) lies in a surface \( S \), then we define the geodesic curvature \( \kappa_g \) of \( \alpha \) to be the magnitude of the projection of the vector \( \kappa N \) into the tangent plane to \( S \).

Let \( n = \frac{S_x \times S_y}{|S_x \times S_y|} \). Then \( n \) is normal to \( S \). As \( \alpha \) lies in \( S \), the vector \( T = \alpha' \) is tangent to \( S \). Therefore, \( n \times T \) is tangent to \( S \), and orthogonal to \( T \). The vector \( \kappa N \) is also orthogonal to \( T \), so we can compute its projection into the tangent plane by

\[
\kappa_g = (\kappa N) \cdot (n \times T).
\]

Since \( T \) is tangent to \( S \), we can express it as a linear combination of \( e_1 \) and \( e_2 \). Thus, there is a function, \( \theta(s) \), such that

\[
T(s) = \cos \theta(s) e_1 + \sin \theta(s) e_2.
\]

The vector \( n \times T \) is thus given by

\[
n \times T = -\sin \theta(s) e_1 + \cos \theta(s) e_2.
\]

We now derive a formula for \( \kappa_g \):
\[ \kappa_g = (\kappa N) \cdot (n \times T) \]
\[ = T' \cdot (-\sin \theta e_1 + \cos \theta e_2) \]
\[ = \frac{d}{ds} (\cos \theta e_1 + \sin \theta e_2) \cdot (-\sin \theta e_1 + \cos \theta e_2) \]
\[ = (\cos \theta e_1' + \sin \theta e_2') \cdot (-\sin \theta e_1 + \cos \theta e_2) \]
\[ + \theta' (-\sin \theta e_1 + \cos \theta e_2) \cdot (-\sin \theta e_1 + \cos \theta e_2) \]
\[ = (\cos \theta e_1' + \sin \theta e_2') \cdot (-\sin \theta e_1 + \cos \theta e_2) + \theta' \]
\[ = -\sin \theta \cos \theta e_1' \cdot e_1 + \sin \theta \cos \theta e_2' \cdot e_2 + \cos^2 \theta e_1' \cdot e_2 - \sin^2 \theta e_2' \cdot e_1 + \theta' \]
\[ = \cos^2 \theta \phi_{12} + \sin^2 \theta \phi_{12} + \theta' \]
\[ = \phi_{12} + \theta' \]

In other words, the geodesic curvature of \( \alpha \) is the sum of the rate of turning of a parallel vector field along \( \alpha \), and the rate of turning of \( T = \alpha' \), both measured with respect to the \( \{e_1, e_2\} \) basis.

By the above equation, \( \theta' = \kappa_g - \phi_{12} \). Suppose \( \alpha \) is now a simple-closed curve in \( S \) bounding a simply-connected region \( R \). Then integrating both sides gives us:

\[ \theta(L) - \theta(0) = \int_{\partial R} \kappa_g \, ds - \int_{\partial R} \phi_{12} \, ds \]
\[ = \int_{\partial R} \kappa_g \, ds + \iint_{R} K \, dA \]

Note that \( \theta(0) \) and \( \theta(L) \) are angles defining the same vector, so \( \theta(0) - \theta(L) \) is a multiple of \( 2\pi \). In fact, this angle difference is precisely \( 2\pi \), as can be seen by continuously shrinking \( R \) to a point. We conclude

\[ 2\pi = \int_{\partial R} \kappa_g \, ds + \iint_{R} K \, dA \]

The curve \( \alpha \) represents a geodesic precisely when \( \kappa_g = 0 \) (by definition!). In this case,

\[ 2\pi = \iint_{R} K \, dA \]

The only convenient example is the top half of a sphere of radius one. This sphere has area \( 4\pi \), and Gaussian curvature \( K = 1 \). Hence, the integral of \( K \) over the top half of the sphere is \( 2\pi \). This agrees with the above formula, since the equator is a geodesic.
LESSON 20
The Gauss-Bonnet Theorem

When $\alpha$ is made up of smaller segments $\alpha_i$, meeting at corners with exterior angles $\epsilon_i$, we can smooth these corners an infinitesimal amount to obtain the following:

$$2\pi = \sum_i \int_{\alpha_i} \kappa_g \, ds + \iint_R K \, dA + \sum_i \epsilon_i$$

When these smaller segments are geodesics, $\kappa_g$ is zero on them. Thus, this simplifies to

$$2\pi = \iint_R K \, dA + \sum_i \epsilon_i$$

When $R$ is a triangle with geodesic edges with interior angles $\iota_i$, we have

$$2\pi = \iint_R K \, dA + \epsilon_1 + \epsilon_2 + \epsilon_3$$

$$= \iint_R K \, dA + (\pi - \iota_1) + (\pi - \iota_2) + (\pi - \iota_3)$$

$$= \iint_R K \, dA + 3\pi - (\iota_1 + \iota_2 + \iota_3)$$

We conclude that the sum of the interior angles of a geodesic triangle is given by

$$\iota_1 + \iota_2 + \iota_3 = \pi + \iint_R K \, dA$$

Note that when $S$ is a plane, $K = 0$, and this agrees with classical Euclidean geometry. When $S$ is a sphere we have seen $K > 0$, and also that the exterior angles of triangles sum to more than $\pi$. In particular, a geodesic triangle on a sphere with vertices at the north pole, and two vertices $1/4$ of the way around the equator will define $1/8$ of the sphere, which has area $4\pi/8 = \pi/2$. Thus, the angles should sum to $3\pi/2$, which indeed they do.

Now suppose we have a surface $S$, and we triangulate it with geodesic triangles $\{\tau^i\}$. Let $V$, $E$, and $F$ be the number of vertices, edges, and faces (triangles) of the triangulation. Then
\[ \int \int_S K \, dA = \sum_i \int \int_{\tau_i} K \, dA = (\text{sum of all interior angles}) - \pi (\text{number of triangles}) = 2\pi V - \pi F \]

Now notice that each triangle has 3 edges, and when \( S \) is closed (i.e. it has no boundary) each edge is contained in 2 triangles. Thus, \( E = 3F/2 \). It follows that

\[ V - E + F = V - 3F/2 + F = V - F/2 \]

and thus,

\[ 2\pi (V - E + F) = 2\pi V - \pi F. \]

The quantity \( V - E + F \) is called the Euler characteristic of \( S \), and is denoted \( \chi(S) \). A fundamental result of topology is that this number is a homeomorphism invariant. In particular, it is independent of the chosen triangulation, and does not change under continuous deformations.

Putting the above equalities together, we obtain the following crowning theorem of Differential Geometry:

**Theorem 20.63 (Gauss-Bonnet).** For any closed surface, \( S \),

\[ \int \int_S K \, dA = 2\pi \chi(S). \]

Thus, the total Gaussian curvature does not change when we deform the surface in a continuous fashion. Making a surface bigger, for example, increases the area over which we might integrate, and must therefore decrease the Gaussian curvature a proportional amount.