FROM THE GOLDEN RATIO TO FIBONACCI PHYLLOTAXIS SPIRALS

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Abstract. We explain why the assumption of growth governed by the Golden Ratio in phyllotaxis leads to a Fibonacci number of spirals. We also explain why two families of spirals in opposite directions are usually evident, both Fibonacci in number. Finally, we explain why flat phyllotaxis patterns, such as in sunflowers, seem to have different numbers of spirals depending on how far from the center you look.

1. Introduction

The Golden Ratio ϕ is the positive number whose square is one larger than itself. The Fibonacci sequence $\{F_n\}$ is defined so that $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$. In almost everything written about the growth of sunflowers and pinecones (i.e. anything governed by *phyllotaxis*), the author notes two facts:

(1) As the plant grows, the angle around the central axis between successive seeds is determined by the Golden Ratio.
(2) If you count the number of apparent spirals, the answer is almost always an element of the Fibonacci sequence, usually either 3, 5, 8, 13, 21, or 34 (although you can certainly find larger numbers, for example, in sunflowers).

The more technical papers on this subject focus on a possible biological mechanism for plant growth, and then give a mathematical argument for why one or the other of these two facts follows as a result. Here we take a different approach and explain why, given the first of these facts, the second is a direct mathematical consequence. Our results also explain why two families of spirals in opposite directions are usually evident, both Fibonacci in number. Finally, we explain why flat phyllotaxis patterns, such as in sunflowers, seem to have different numbers of spirals depending on how far from the center you look.

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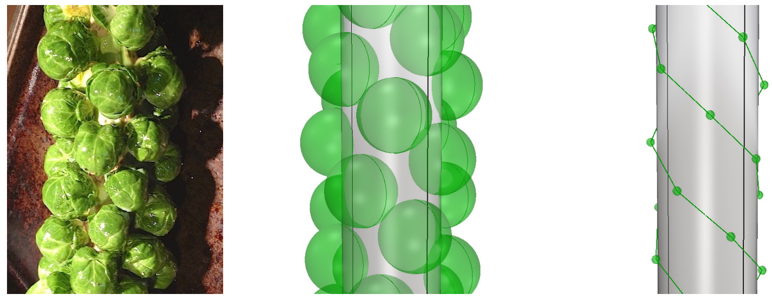


(a) (b) (c)

Figure 1. (a) Brussels Sprout stalk. (b) Cylindrical phyllotaxis model. (c) Spirals created by connecting each point to its two closest neighbors.

2. Cylindrical Phyllotaxis

We'll start with cylindrical examples of phyllotaxis, like in the Brussels sprout stalk shown in Figure 1(a). To create the model shown in Figure 1(b), we place our "sprouts" at points around a vertical cylinder as follows. Begin at the bottom and place the first point. Then rotate around the cylinder $1/\phi$ of its circumference (i.e. an angle of $2\pi/\phi$ radians or $360/\phi$ degrees), and move up the cylinder a distance h. Place the second point there, and repeat.

To our eye, there are obvious spirals apparent in Figure 1(b). The explanation for these perceived spirals is simple. In Figure 1(c) we show the result of connecting each point to its two closest neighbors by line segments. For visualization, we've made the size of each "sprout" in this image smaller. Note that the resulting curves are precisely the spirals that you see in Figure 1(b). In other words, the spirals you see in Figure 1(b) are just due to the fact that your eye is picking out the points that are closest together.

Let's label the points in the order they were added, starting from zero. As we go around the stalk adding seeds we eventually come to a point that is closer to point 0 than any other seed will be. The distance from point 0 to point n is the same as the distance from point n to point $2n$, and the distance from $2n$ to $3n$. Visually, these points all lie on the same spiral. For example, in Figure 2, point 0 is connected to point 8 and point 8 is also connected to point 16, etc.

Note that the distance from point 0 to point n is the same as the distance from point 1 to point $n + 1$. Hence, there is another spiral going through points 1, $n + 1$ $2n + 1$, etc. Similarly, there is a third

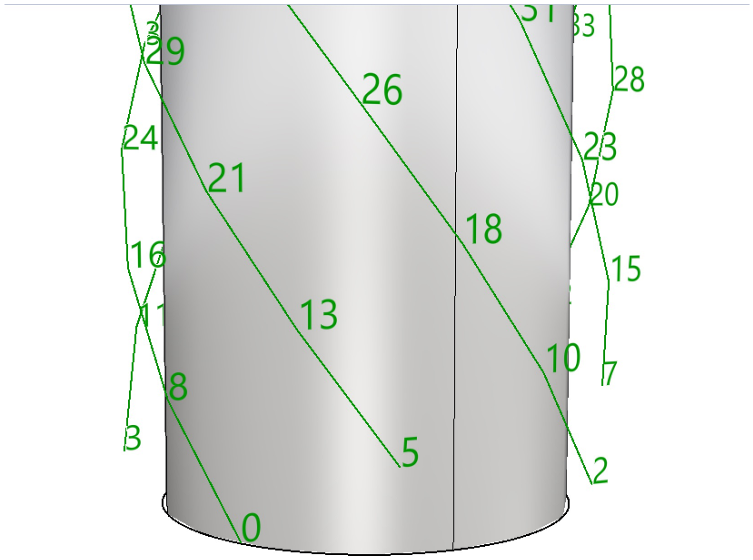


FIGURE 2. Numbering the points in the order they were added. If point 0 is closest to point 8, then there will be 8 spirals.

spiral going through points 2, $n + 2$, $2n + 2$, etc. Continuing in this way, we find spirals that also start at points 3, 4, 5, all the way up to $n - 1$. Since the first spiral started at point 0, we must have a total of n spirals. See Figure 2. This observation is worth repeating: if point n is the closest to point 0, then there will be n spirals. Hence, if we are interested in the number of spirals, we have to get a handle on which point is closest to point 0.

To locate a point on a cylinder relative to point 0, we rotate around the cylinder some amount and go up some amount. If we do more than one full rotation, then it may be shorter to rotate a smaller amount the other way. We will call the smallest amount you have to rotate to get from point 0 to point i the *net rotation*. We will measure net rotation as a fraction of a full rotation, so that it will always be between $-1/2$ and $1/2$ (with the sign determined by which direction you have to rotate). For example, if some point is constructed by doing 5.83 rotations from point 0, then the net rotation of that point is -0.17. Similarly, if we do 6.27 rotations to get from point 0 to point i, then the net rotation of point i will be 0.27. The net rotation of the ith point, which we denote $\omega(i)$, is easily calculated by subtracting from i/ϕ the nearest integer.

If we assume that units have been chosen so that the circumference of the cylinder is one, then to get from point 0 to point i by the shortest path along the cylinder, you go around the cylinder by $\omega(i)$ and up by ih. The distance between the two, $D(i)$, is given by the Pythagorean theorem:

$$D(i) = \sqrt{[\omega(i)]^2 + (ih)^2}$$

Observe the following about the function $D(i)$:

(1) As h goes to zero, $D(i)$ tends toward $|\omega(i)|$. Thus, as we shrink h, a point with smaller net rotation than all of the ones below it eventually becomes closer to point 0 than they are.
(2) If the net rotation of any point higher than point i is greater than that of point i, then it will never be closer to point 0 than point i.

We now state our main technical result, which we prove in Section 5.

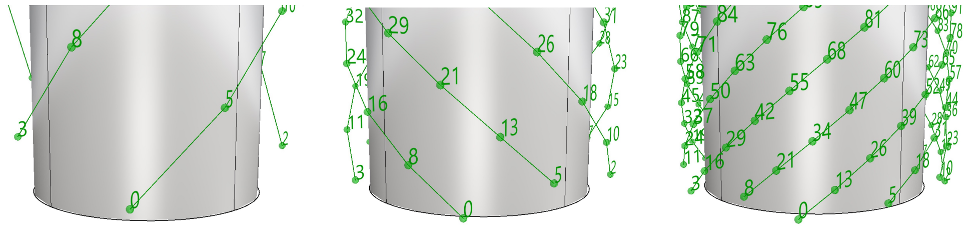
Theorem 1. *For each Fibonacci number F_n,*

$$|\omega(F_n)| = \min\{|\omega(i)| \textit{ for } i < F_{n+1}\}.$$

This theorem says that each point corresponding to a Fibonacci number will have the smallest net rotation among all points lower than the point corresponding to the next Fibonacci number. This immediately implies that the net rotation of point F_5 (for example) is smaller than the net rotations of any point below it, so by the first observation above as we shrink h this point will become closest to point 0. However, Theorem 1 also says that the net rotation of point F_5 is smaller than that of all of the points between it and point F_6. Thus, by the second observation above as we shrink h further the point F_5 will continue to be closest to point 0, until the point F_6 becomes closest. In other words, as we shrink h each point corresponding to a Fibonacci number will become closest to point 0, until the point corresponding to the next Fibonacci number becomes closest. See Figure 3. As the number of the closest point determines the number of spirals, the number of spirals is always a Fibonacci number!

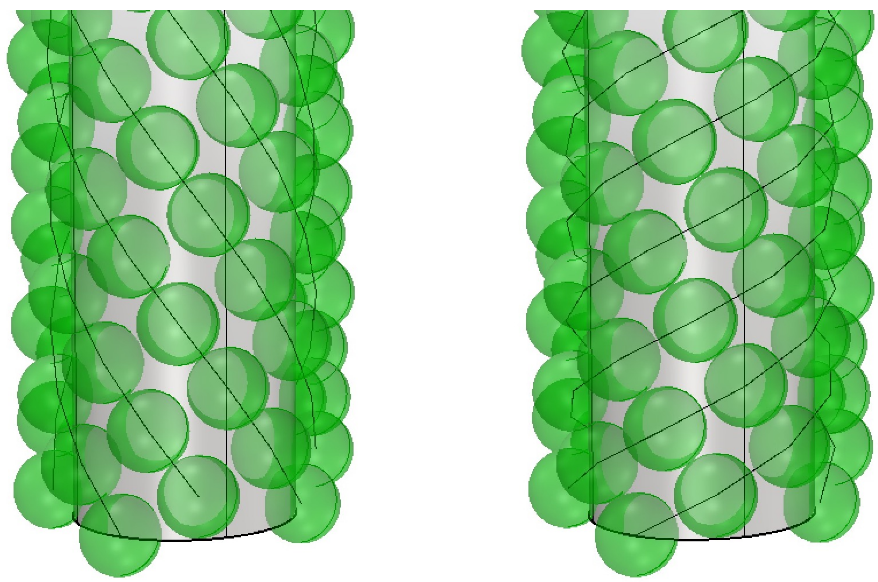
3. Minor spirals

Often when we look at objects that exhibit phyllotaxis, we actually see two sets of spirals. The set that is most visually dominant we'll call the *major spirals*. These are the ones described in the previous section, coming from connecting each point to its closest neighbors. The other set, which we'll call the *minor spirals*, comes from connecting each



(a) (b) (c)

FIGURE 3. As we shrink h (the vertical spacing), the number of spirals changes.



(a) (b)

FIGURE 4. (a) Major Spirals. (b) Minor Spirals.

point to its second closest neighbors. See Figure 4. Theorem 1 (and Lemma 6, to be presented later) explains this set as well.

It follows from Theorem 1 that if point F_n is the closest neighbor to point 0, then point F_{n-1} is the second closest. By reasoning identical to that of the previous section, this leads us to conclude that there are F_{n-1} minor spirals. Furthermore, we will see in Lemma 6 that the signs of $\omega(F_n)$ and $\omega(F_{n-1})$ are opposite. This tells us that the major and minor spirals must be in opposite directions, with one set

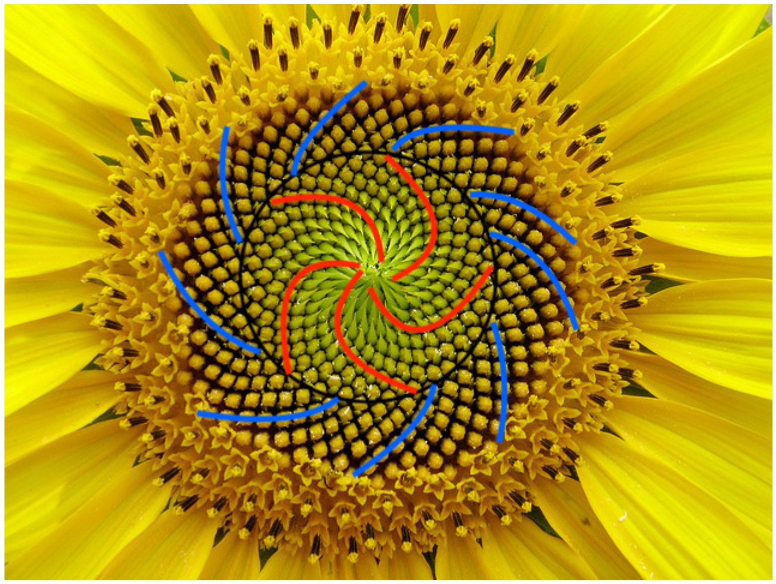


FIGURE 5. Near the center of the sunflower our eye picks up 21 spirals, some of which are shown in red. Toward the outside the 34 spirals in the opposite direction are more visually dominant, shown in blue.

spiraling clockwise around the cylinder, and the other spiraling counter-clockwise.

4. Sunflowers

Now that we have a better understanding of cylindrical phyllotaxis spirals we are ready to take on flat phyllotaxis patterns, like the sunflower shown Figure 5. In these patterns, we tend to see a different number of spirals depending on how far away from the center you focus. Regardless, these numbers are still usually Fibonacci.

In the previous section we looked at points on a cylinder determined only by a vertical spacing, h. The units for h were chosen so that the circumference of the cylinder is one. Thus, keeping the vertical spacing constant but enlarging the radius of the cylinder has the same effect on the numerical value of h. Hence, all of our reasoning is still valid in this scenario: enlarging the radius of the cylinder will increase the number of spirals, and this number will always be Fibonacci. Of course, simultaneously enlarging the radius of the cylinder and shrinking the vertical distance between consecutive points will only accelerate the change in the number of spirals.

Now consider a flat phyllotaxis picture, and let A be the portion of this picture between concentric circles of radii r_1 and r_2. As before, the (major) spirals you see in A are determined by the closest neighbors of each point. Let h be the average radial component of the distance between consecutive points in A. If r_1 is close to r_2, then the distance between the points in A will be comparable to the distance between phyllotaxis points on a cylinder of radius $\frac{r_1+r_2}{2}$, and vertical spacing h between consecutive points. Thus, the number of apparent spirals must be Fibonacci.

Further out from the center of the picture $\frac{r_1+r_2}{2}$ will be larger, and the radial component of the distance between consecutive points tends to be smaller. Thus, further away from the center you'll see a picture more like that on a cylinder with a larger number of spirals.

As noted in the previous section, Lemma 6 tells us that the net rotations of consecutive Fibonacci numbers have opposite sign. Hence, as we look further from the center of the sunflower not only do the number of spirals increase, but they switch direction as they do so. See Figure 5.

5. Proof of Theorem 1.

In this section we present a series of technical results about net rotation number, building up to a proof of Theorem 1. Much of this material can likely be obtained using known results about Diophantine approximations of the Golden ratio. The proofs given here are more basic.

Lemma 2. *If* $|\omega(n) + \omega(m)| < 1/2$*, then* $\omega(n + m) = \omega(n) + \omega(m)$*.*

Proof. Imagine stacking the portion of the cylinder with points 0 through m on top of the portion with points 0 through n, and rotating so that the bottom point of the upper cylinder is directly above the top point of the lower one. Then the net rotation from the bottom point of the bottom cylinder to top point of the top cylinder will be the sum of the two. If the total is less than 1/2, then it will equal the net rotation. □

Lemma 3. *If* $|\omega(n)|$ *and* $|\omega(m)|$ *are less than* $1/4$*, then* $\omega(n - m) = \omega(n) - \omega(m)$*.*

Proof. Without loss of generality, assume $\omega(n - m) + \omega(m)$ is positive. If $\omega(n-m)+\omega(m) < 1/2$ then the result follows from Lemma 2. If not, then stacking cylinders as described in the proof of Lemma 2 results in a total rotation over 1/2. As $|\omega(m)| < 1/4$, it must be the case that $\omega(n - m) > 0$. Since $\omega(n - m) < 1/2$ and $|\omega(m)| < 1/4$, the total rotation is less than 1. Hence, the net rotation is obtained by subtracting the sum from 1. That is,

$$\omega(n) = 1 - (\omega(n-m) + \omega(m)).$$

However, since $\omega(n-m) < 1/2$, we have

$$\begin{aligned}\omega(n) &= 1 - \omega(n-m) - \omega(m) \\ &> 1 - 1/2 - \omega(m) \\ &= 1/2 - \omega(m) \\ &> 1/4\end{aligned}$$

We have thus reached a contradiction. □

Lemma 4. $\omega(1) = \frac{1}{\phi} - 1$ *and* $\omega(2) = \frac{2}{\phi} - 1$.

Proof. By construction, to go from point 0 to point 1, we rotate around the cylinder $1/\phi$ of a full rotation. As $1/\phi$ is between 1/2 and 1, the net rotation is given by $\frac{1}{\phi} - 1$. Similarly, to get from point 0 to point 2 we do a total rotation of $2/\phi$. As $2/\phi$ is between 1 and 3/2, the net rotation is $\omega(2) = \frac{2}{\phi} - 1$. □

Lemma 5. $\omega(F_n) = \frac{F_n}{\phi} - F_{n-1}$

The proof is an easy induction argument using Lemma 4 for the base cases and Lemma 2 for the inductive step.

Lemma 6. $\{\omega(F_n)\}$ *is an alternating sequence whose absolute values converge monotonically to 0.*

Proof. The following is a well-known closed-form expression for F_n:

$$F_n = \frac{1}{\sqrt{5}}\left(\phi^n - \frac{(-1)^n}{\phi^n}\right)$$

Combining this with the formula given for $\omega(F_n)$ in Lemma 5 and applying a little algebra gives us

$$\omega(F_n) = \frac{(-1)^{n+1}}{\sqrt{5}}\left(\frac{1}{\phi^{n+1}} + \frac{1}{\phi^{n-1}}\right)$$

The result follows. □

We are now prepared to give the proof of Theorem 1.

Proof. Choose n minimal so that the theorem is false, and let i be an integer so that $|\omega(i)| \le |\omega(j)|$ for all $j < F_{n+1}$. Since we are assuming the theorem is false, $|\omega(i)| < |\omega(F_n)|$. By Lemma 6, i can't be a Fibonacci number, because any Fibonacci number less than F_n would have absolute net rotation larger than $|\omega(F_n)|$. Hence, i is at least

4 (the smallest non-Fibonacci number), and thus $F_{n+1} \geq 5$, making $F_n \geq 3$. Direct calculation using Lemma 5 shows $\omega(3) < 1/4$. It now follows from Lemma 6 that $|\omega(F_n)| < |\omega(3)| < 1/4$. Thus, $|\omega(i)| < 1/4$.

We claim $F_n < i$. If not, then let m be the largest integer so that $F_m < i$, and thus $i < F_{m+1}$. We know (by assumption) $|\omega(i)| < |\omega(F_n)|$ and by Lemma 6 we have $|\omega(F_n)| < |\omega(F_m)|$. Thus, i is a number less than F_{m+1} with $|\omega(i)| < |\omega(F_m)|$. The theorem thus fails for the number $m < n$, contradicting the minimality of our choice of n.

The proof now breaks up into two cases. In the first case, $\omega(i)$ has the same sign as $\omega(F_n)$. Since both $|\omega(i)|$ and $|\omega(F_n)|$ are less than $1/4$, Lemma 3 implies $\omega(i - F_n) = \omega(i) - \omega(F_n)$. Since they have the same sign, and $|\omega(i)| < |\omega(F_n)|$, then

$$|\omega(i - F_n)| = |\omega(i) - \omega(F_n)| < |\omega(F_n)| < |\omega(F_{n-1})|.$$

Since $i < F_{n+1}$, we have $i - F_n < F_{n+1} - F_n = F_{n-1}$. In summary, we have shown $i - F_n$ is a number less than F_{n-1} with $|\omega(i - F_n)| < |\omega(F_{n-1})|$. Hence, the theorem fails for $n - 1$, contradicting our minimality assumption on n.

In the second case, $\omega(i)$ has opposite sign as $\omega(F_n)$. By Lemma 6 $\omega(F_{n+1})$ and $\omega(F_n)$ also have opposite sign, and therefore $\omega(i)$ has the same sign as $\omega(F_{n+1})$. By Lemma 3, $\omega(F_{n+1} - i) = \omega(F_{n+1}) - \omega(i)$. Since they have the same sign and both have smaller magnitude than $|\omega(F_n)|$,

$$|\omega(F_{n+1} - i)| = |\omega(F_{n+1}) - \omega(i)| < |\omega(F_n)| < |\omega(F_{n-1})|.$$

At the beginning of this proof we established $F_n < i$. It thus follows that $F_{n+1} - i < F_{n+1} - F_n = F_{n-1}$. Hence, as in the previous case the theorem fails for $n - 1$, contradicting our minimality assumption on n. □