TOPOLOGICAL INDEX THEORY FOR SURFACES IN 3-MANIFOLDS

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Abstract. The disk complex of a surface in a 3-manifold is used to define its topological index. Surfaces with well-defined topological index are shown to generalize well-known classes, such as incompressible, strongly irreducible, and critical surfaces. The main result is that one may always isotope a surface \( H \) with topological index \( n \) to meet an incompressible surface \( F \) so that the sum of the indices of the components of \( H \setminus N(F) \) is at most \( n \). This theorem and its corollaries generalize many known results about surfaces in 3-manifolds, and often provides more efficient proofs. The paper concludes with a list of questions and conjectures, including a natural generalization of Hempel’s distance to surfaces with topological index \( \geq 2 \).

1. Introduction.

Let \( H \) be a properly embedded, separating surface with no torus components in a compact, orientable 3-manifold \( M \). Then the disk complex, \( \Gamma(H) \), is defined as follows:

1. Vertices of \( \Gamma(H) \) are isotopy classes of compressions for \( H \).
2. A set of \( m + 1 \) vertices forms an \( m \)-simplex if there are representatives for each that are pairwise disjoint.

Here we explore what information is contained in the topology of \( \Gamma(H) \). To this end, we define

Definition 1.1. The homotopy index of a complex \( \Gamma \) is defined to be 0 if \( \Gamma = \emptyset \), and the smallest \( n \) such that \( \pi_{n-1}(\Gamma) \) is non-trivial, otherwise. We say the topological index of a surface \( H \) is the homotopy index of its disk complex, \( \Gamma(H) \). If \( H \) has a topological index then we say it is topologically minimal.

When \( H \) is the boundary of a handlebody then the disk complex was first defined by McCullough in [McC91], who showed that in this case \( \Gamma(H) \) is contractible. It follows that such surfaces are not topologically minimal. The goal of the present paper is to show that topologically

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minimal surfaces are a natural generalization of several well-known classes of surfaces in 3-manifolds, and that the results that hold for each of these classes also hold true for all topologically minimal surfaces. As an added benefit, proofs involving the set of all topologically minimal surfaces are often much shorter than existing proofs involving just, say, index 2 surfaces. This is largely owing to the inductive nature of the arguments.

By definition, incompressible surfaces have topological index 0. In the next section we show that the strongly irreducible surfaces of Casson and Gordon [CG87] are precisely those that have topological index 1. We also show that critical surfaces, previously defined by the author in [Bac02] and [Bac08], have topological index 2. One important property shared by these types of surfaces is that they may always be isotoped to meet an incompressible surface in a collection of loops that are essential on both. We show here that this is in fact a corollary of a powerful result about all topologically minimal surfaces. This is given by Theorem 3.8, which asserts that a topologically minimal surface $H$ and an incompressible surface $F$ can be isotoped so that $H \setminus N(F)$ is topologically minimal in $M \setminus N(F)$.

Section 4 contains corollaries to Theorem 3.8. We show there that if $M$ contains a topologically minimal Heegaard surface then $\partial M$ is incompressible. It then follows that if a closed 3-manifold $M$ contains any topologically minimal surface $H$ then either it is a Heegaard surface, $M$ is Haken, or $H$ is contained in a ball. (In the final section we conjecture that this last possibility can not happen.) Finally, we show that if the disjoint union of surfaces is topologically minimal then so are its components, and its topological index is the sum of the indices of its components. Combining this with Theorem 3.8, we find that a surface $H$ with topological index $n$ can be isotoped to meet an incompressible surface $F$ in such a way so that the sum of the indices of the components of $H \setminus N(F)$ is at most $n$. This is a generalization of known results about topological index 0 and 1 surfaces.

In any new theory, the questions raised are as important as the new results. In the final section of this paper we list a few tantalizing questions and conjectures about topologically minimal surfaces. These include conjectures about the possible indices of topologically minimal surfaces in various kinds of 3-manifolds, a natural generalization of Hempel’s distance invariant [Hem01] to surfaces of arbitrary topological index, and conjectures which relate geometric minimal surfaces to topologically minimal surfaces.

Much of the motivation for this work comes from ideas of Hyam Rubinstein. In the late 1990’s Rubinstein pioneered the viewpoint that
strongly irreducible Heegaard splittings were the right class of surfaces within which to search for unstable (geometrically) minimal surfaces of index 1, as well as their PL analogues, the so-called “almost normal” surfaces. One often finds such surfaces by minimax arguments involving 1-parameter sweepouts. Many of the topological arguments involving strongly irreducible surfaces also use 1-parameter sweepouts, so it became natural to think about such surfaces as being “topologically minimal,” in a very imprecise sense. In later work the author defined critical surfaces as an attempt to find some topological analogue to geometrically minimal surfaces that have index 2. As one would expect from such an analogue, arguments involving critical surfaces often involve 2-parameter sweepouts. In this paper we make precise the idea of topological index, demonstrate its usefulness, and conjecture its relation to geometric minimal surfaces.

In [Baca] various relative versions of topological index are given. This allows us, for example, to show that topologically minimal surfaces can be isotoped into a suitably nice position with respect to a triangulation, analogous to Kneser [Kne29] and Haken’s [Hak68] normal surfaces, and Rubinstein’s [Rub95] almost normal surfaces. It then follows from various recent results of the author, Derby-Talbot, Jaco, Rubinstein, and Sedgwick that complicated amalgamating surfaces act as barriers to low index, low genus, topologically minimal surfaces. This is the key technical tool necessary for the author’s construction of a counter-example to the Stabilization Conjecture for Heegaard splittings, as well as several results about amalgamation and isotopy of Heegaard splittings. For a preview of these results, see [Bacb].

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2. Low index surfaces

In this section we show that the concept of topological index generalizes several well known classes of surfaces in 3-manifolds.
Definition 2.1. Let $H$ be a properly embedded surface in a 3-manifold $M$. A loop $\alpha$ on $H$ is essential if it does not bound a subdisk of $H$. A disk $D$ is a compression for $H$ if $D \cap H = \partial D$ is an essential loop on $H$. The surface $H$ is incompressible if there are no compressions for it. If $D$ is a compression for $H$ then we construct the surface $H/D$ as follows. Let $M(H)$ denote the manifold obtained from $M$ by cutting open along $H$. Let $B$ denote a neighborhood of $D$ in $M(H)$. The surface $H/D$ is obtained from $H$ by removing $B \cap H$ and replacing it with the frontier of $B$ in $M(H)$.

It follows immediately from the definitions that a surface has topological index 0 if and only if it is incompressible. We now show that surfaces with topological index 1 and 2 are also familiar.

Let $\mathcal{V}$ and $\mathcal{W}$ denote the sides of a Heegaard surface $H$, and $\Gamma_\mathcal{V}(H)$ and $\Gamma_\mathcal{W}(H)$ the subspaces of $\Gamma(H)$ spanned by compressions in $\mathcal{V}$ and $\mathcal{W}$. McCullough has called these complexes the disk complexes of $\mathcal{V}$ and $\mathcal{W}$. McCullough proved that such disk complexes are contractible [McC91]. It follows that the topology of $\Gamma(H)$ is entirely determined by the simplices that connect $\Gamma_\mathcal{V}(H)$ to $\Gamma_\mathcal{W}(H)$. With this in mind, it is natural to introduce special terminology when there are no edges connecting $\Gamma_\mathcal{V}(H)$ to $\Gamma_\mathcal{W}(H)$. The following definition is due to Casson and Gordon [CG87].

Definition 2.2. $H$ is strongly irreducible if there are compressions on opposite sides of $H$, but each compression on one side meets all compressions on the other.

The main result of [CG87] is that if the minimal genus Heegaard splitting of a 3-manifold is not strongly irreducible, then the manifold contains an incompressible surface.

Theorem 2.3. $H$ has topological index 1 if and only if it is strongly irreducible.

Proof. By definition, a surface has topological index 1 when $\pi_0(\Gamma(H))$ is non-trivial. Hence, in this case $\Gamma(H)$ is disconnected. However, by McCullough’s result $\Gamma_\mathcal{V}(H)$ and $\Gamma_\mathcal{W}(H)$ are contractible, so the only way for $\Gamma(H)$ to be disconnected is if both $\Gamma_\mathcal{V}(H)$ and $\Gamma_\mathcal{W}(H)$ are non-empty, and there are no edges connecting them. There are thus compressions on both sides, but any pair of such compressions intersect. \qed

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contains an incompressible surface. Critical surfaces were also instrumental in the author’s proof of a conjecture of C. Gordon [Bac08].

**Definition 2.4.** $H$ is critical if the compressions for $H$ can be partitioned into sets $C_0$ and $C_1$ such that:

1. For each $i = 0, 1$ there is at least one pair of disks $V_i, W_i \in C_i$ on opposite sides of $H$ such that $V_i \cap W_i = \emptyset$.
2. If $V \in C_0$ and $W \in C_1$ are on opposite sides of $H$ then $V \cap W \neq \emptyset$.

**Theorem 2.5.** $H$ has topological index 2 if and only if it is critical.

**Proof.** We first establish that if $H$ has topological index 2 then it is critical. Let $\Gamma_{VW}(H)$ be the subspace of $\Gamma(H)$ consisting of those cells spanned by vertices in both $\Gamma_V(H)$ and $\Gamma_W(H)$. Since $\Gamma_V(H)$ and $\Gamma_W(H)$ are contractible and $\pi_1(\Gamma(H)) \neq 1$, there is a non-trivial loop in $\Gamma(H)$ that passes from $\Gamma_V(H)$ to $\Gamma_W(H)$ and back, crossing through $\Gamma_{VW}(H)$ exactly twice. The two edges $(V_0, W_0)$ and $(V_1, W_1)$ of $\Gamma_{VW}(H)$ traversed by this path must be in different components of $\Gamma_{VW}(H)$, and thus we conclude $\Gamma_{VW}(H)$ is disconnected. We may therefore partition the components of $\Gamma_{VW}(H)$ into two non-empty sets, $C_0$ and $C_1$, where $(V_i, W_i) \subset C_i$. Since $C_0$ and $C_1$ are a partition of the components of $\Gamma_{VW}(H)$, there are no edges $(V, W)$ that connect them, were $V \in C_0$ and $W \in C_1$. Note that any vertex of $\Gamma(H)$ that is not in $\Gamma_{VW}(H)$ can be added to either $C_0$ or $C_1$, and the conditions of Definition 2.4 will still be satisfied.

We now prove that if $H$ is critical then it has topological index 2. Let $C_i, V_i,$ and $W_i$ be as in Definition 2.4. We must produce a non-trivial loop in $\Gamma(H)$. Since $\Gamma_V(H)$ is contractible, there is a path of compressions in $\Gamma_V(H)$ from $V_0$ to $V_1$. Similarly, there is a path from $W_0$ to $W_1$ in $\Gamma_W(H)$. These two paths, together with the edges $(V_i, W_i)$, form a loop $\alpha$ in $\Gamma(H)$. By way of contradiction, suppose $\alpha$ is trivial in $\pi_1(\Gamma(H))$. Then there is a map $f$ of a disk $D$ into $\Gamma(H)$ such that $f(\partial D) = \alpha$. For some triangulation $T$ of $D$, we may assume $f$ is simplicial. We now assume that all choices have been made so that the number of 2-simplices in $T$ is minimal.

Let $\Delta$ denote the triangle in $T$ that has $(V_0, W_0)$ as one of its edges. Without loss of generality we assume the third vertex of $\Delta$ represents a compression in $\mathcal{V}$, and denote it as $V$. Since $(V, W_0)$ is an edge of $\Delta$, it follows that $V \cap W_0 = \emptyset$. Hence, by criticality $V \in C_0$. If $V$ is in the interior of $D$ then remove $\Delta$ from $D$ and replace $V_0$ with $V$. This increases the combinatorial length of $\partial D$, but reduces the number of 2-simplices in $T$, a contradiction.
The remaining case is when $V$ is in $\partial D$. Then the edge $(V,W_0)$ cuts $D$ into two smaller disks. One of these, $D'$, contains the edge $(V_1,W_1)$. If we now replace $D$ with $D'$ and $V_0$ with $V$, we again contradict our minimality assumption. □

3. Topological index in the complement of a surface

In this section we show that a topologically minimal surface can always be isotoped so that it meets the complement of an incompressible surface in a topologically minimal surface.

Definition 3.1. Let $H$ and $F$ be properly embedded surfaces in a 3-manifold $M$. Let $D$ be a compression for $H$. We say $D$ has a shadow (with respect to $F$) if there is a disk $D'$ where $\partial D' = \partial D$, $D' \cap F = \emptyset$, and the interior of $D'$ meets $H$ in loops that are inessential on $H$. The disk $D'$ is said to be the shadow of $D$. See Figure 1.

The main focus of this paper is to find relationships between the homotopy indices of various complexes that depend on a specific position of $H$ with respect to an incompressible surface $F$. The first of these is the disk complex $\Gamma(H)$ of $H$. The complex $\Gamma_F(H)$ is the subset of $\Gamma(H)$ such that each vertex has a shadow. Later we will encounter a third complex, $\Gamma(H^F)$.

The relationship between the homotopy indices of the complexes $\Gamma(H)$ and $\Gamma_F(H)$ is given presently in Theorem 3.2. In subsequent sections we will use this theorem to prove that when $H$ is topologically minimal, then it can be isotoped so that it is topologically minimal in the complement of $F$. We then show that many of the standard results in 3-manifold topology, presently known for surfaces with low topological index, generalize to surfaces with arbitrary topological index.
Theorem 3.2. Let \( H \) and \( F \) be properly embedded surfaces in \( M \), where \( H \) has topological index \( n \). Then \( H \) may be isotoped so that

1. \( H \) meets \( F \) in \( p \) points of tangency, for some \( p \leq n \). Away from these tangencies \( H \) is transverse to \( F \).
2. The complex \( \Gamma_{F}(H) \) has homotopy index \( i \leq n - p \).

Proof. When \( H \) has topological index 0 the result is immediate, as \( \Gamma_{F}(H) \subset \Gamma(H) = \emptyset \). We will assume, then, that \( H \) has topological index \( n \geq 1 \). It follows that \( \pi_{n-1}(\Gamma(H)) \) is non-trivial, and thus there is a map \( \iota : S \to \Gamma(H) \) of an \((n-1)\)-sphere \( S \) into the \((n-1)\)-skeleton of \( \Gamma(H) \) which is not homotopic to a point. Let \( B \) be the cone on \( S \) to a point \( z \). (The point \( z \) is necessarily not in \( \Gamma(H) \).) Hence, \( B \) is an \( n \)-ball.

Our first challenge is to define a continuous family of surfaces \( H_{x} \) in \( M \) isotopic to \( H \), where \( x \in B \). Let \( T \) be a triangulation of \( S = \partial B \) so that the map \( \iota \) is simplicial. Let \( \{v_{i}\} \) denote the set of vertices of \( \Gamma(H) \) that are contained in \( \iota(S) \). For each \( i \) choose a representative \( D_{i} \) from the isotopy class of disks represented by \( v_{i} \) so that if \((v_{i}, v_{j})\) is an edge of \( \Gamma(H) \), then \( D_{i} \cap D_{j} = \emptyset \). For each \( i \), let \( N_{i} \) be a small enough neighborhood of \( D_{i} \) in \( M \) so that \( N_{i} \cap N_{j} = \emptyset \) whenever \((v_{i}, v_{j})\) is an edge of \( \Gamma(H) \). If \( D_{i} \) is a compressing disk for \( H \) then let \( f_{i} \) be a homeomorphism that takes \( N_{i} \) to the standard unit ball in \( \mathbb{R}^{3} \). Choose \( f_{i} \) so that \( f_{i}(H \cap N_{i}) \) is the graph of \( r = 1 \) (in cylindrical coordinates), and \( f_{i}(D_{i}) \) is a disk in the \( xy \)-plane. For each disk \( D_{i} \) we now define a family of surfaces \( H_{i}(t) \) in \( N_{i} \), parameterized by a variable \( t \in [0, 1] \). These surfaces are given by the images of the graphs of \( r = tz^{2} + 1 - t \), under the map \( f_{i}^{-1} \) (see Figure 2).

Extend \( T \) to a triangulation \( T' \) on \( B \) by coning each simplex of \( T \) to the point \( z \). Suppose \( \{D_{0}, ..., D_{n-1}\} \) is the image of an \((n-1)\)-simplex \( \Delta \) of \( T \) under the map \( \iota \). We now identify the \( n \)-simplex of \( T' \) which is the cone on \( \Delta \) with the unit cube in \( \mathbb{R}^{n} \). Label the axes of \( \mathbb{R}^{n} \) with the variables \( t_{0}, ..., t_{n-1} \). Place \( z \) at the origin, and the vertex \( v \) of \( \Delta \) such that \( \iota(v) = D_{i} \) at the point with \( t_{i} = 1 \) and \( t_{j} = 0 \) for all \( j \neq i \).
Figure 3. A simplex $\Delta$ of $T'$, and a few of the surfaces $H_x$ for $x \in \Delta$. The union of the faces of the cube that do not meet $z$ is a simplex of $T$.

If $p$ is at the barycenter of a face $\sigma$ of $\Delta$ then place it at the vertex of the cube where the coordinates corresponding to the vertices of $\sigma$ are 1 and the other coordinates are 0. We now linearly extend over the entire simplex to complete the identification with the cube. Now, if $x$ is in this $n$-simplex then $x$ has coordinates $(t_0(x), \ldots, t_{n-1}(x))$. Let $H_x$ be the surface obtained from $H$ by replacing $H \cap N_i$ with the surface $H_i(t_i(x))$, for each $i$ between 0 and $n-1$. See Figure 3. Repeating this for each $n$-simplex of $T'$ gives us the complete family of surfaces $H_x$.

We assume $H$ is initially transverse to $F$. For each $i$, the surface $H_i(t) \subset N_i$ is tangent to $F$ for finitely many values $\{t_i^j\}$ of $t$. Hence, for each $x \in B$ the surface $H_x$ is tangent to $F$ at finitely points, and each such point is in a distinct ball $N_i$. Note also that if $t_i(x) = t_i(y)$, then $H_x$ and $H_y$ agree inside of $N_i$. Hence, if $H_x$ is tangent to $F$ in $N_i$ then the surface $H_y$ will also be tangent to $F$, for all $y$ in the plane where $t_i(y) = t_i(x)$. It follows that each $n$-simplex of $T'$ is cubed by the points $x$ where $H_x$ is tangent to $F$. See Figure 4. Hence, $B$ is cubed by the $n$-simplices of $T'$, together with this cubing of each such simplex. We denote this cubing of $B$ as $\Sigma$. It follows that if $x$ is in a
codimension $p$ cell of $\Sigma$ then the surface $H_x$ is tangent to $F$ in at most $p$ points.

We now produce a contradiction by defining a continuous map $\Psi$ from $B$ into $\Gamma(H)$. The map $\Psi|\partial B$ will be equal to $\iota$ on the barycenters of the $(n-1)$-cells of $T$, which will in turn imply that $\Psi$ maps $S$ onto $\iota(S)$ with the same degree as $\iota$. A contradiction follows as $\iota(S)$ is not homotopic to a point.

For each $x \in B$ let $V_x = \Gamma_F(H_x)$. If $\tau$ is a cell of $\Sigma$, then we define $V_x$ to be the set $V_x$, for any choice of $x$ in the interior of $\tau$. Note that if $x$ and $y$ are in the interior of the same cell $\tau$ of $\Sigma$, then the pair $(H_x, F)$ is isotopic to $(H_y, F)$. Hence, $V_x = V_y$, and thus $V_\tau$ is well defined.

The map $\Psi$ defined below will take each cell $\tau$ of $\Sigma$ into $V_\tau$. First, we establish a few properties of $V_\tau$.

**Claim 3.3.** Suppose $\sigma$ is a cell of $\Sigma$ which lies on the boundary of a cell $\tau$. Then $V_\sigma \subset V_\tau$.

**Proof.** Pick $x \in \sigma$ and $y \in \tau$. If $D \in V_x$ then $D$ is isotopic to a compression for $H_x$ that has a shadow $D'$. To show $D \in V_y$ we must show that $D$ is isotopic to a compression for $H_y$ that has a shadow. Note that $H_y \cap F$ is obtained from $H_x \cap F$ by resolving some tangency. Hence, any loop of $H_x \setminus F$ is isotopic to a loop of $H_y \setminus F$. It follows that since $D \cap H_x = D' \cap H_x$ was a loop on $H_x$ disjoint from $F$, then $D' \cap H_y$ will be a loop on $H_y$ that is disjoint from $F$. Furthermore, as

![Figure 4. A simplex $\Delta$ of $T'$ is cut up by planes into subcubes. Each such plane is determined by the points $x$ in which $H_x$ is tangent to $F$ in $N_i$, for some $i$.](image-url)
the interior of $D'$ meets $H_x$ in a collection of loops that are inessential on $H_x$, it follows that the interior of $D'$ meets $H_y$ in a collection of loops that are inessential on $H_y$. We conclude $D'$ is a shadow for $D$, both as a compression for $H_x$ and as a compression for $H_y$. Hence, $D \in \Gamma_F(H_y) = V_y$. 

\textbf{Claim 3.4.} For each cell $\tau$ of $\Sigma$, 

$$\pi_i(V_\tau) = 1 \text{ for all } i \leq \dim(\tau) - 1.$$ 

\textit{Proof.} Let $x$ be in the interior of a codimension $p$ cell $\tau$ of $\Sigma$. Then the dimension $\dim(\tau)$ is $n - p$. The surface $H_x$ is tangent to $F$ in at most $p$ points, and is transverse to $H_x$ elsewhere. Recall $V_x = \Gamma_F(H_x)$. Thus, if the theorem is false then $V_x$ is non-empty, and $\pi_i(V_x) = 1$ for all $i \leq n - p - 1 = \dim(\tau) - 1$. \hfill \square

We now define $\Psi$ on the 0-skeleton of $\Sigma$. For each 0-cell $x \in \Sigma$, we will choose a point in $V_x$ to be $\Psi(x)$. If $x$ is in the interior of $B$ then $\Psi(x)$ may be chosen to be an arbitrary point of $V_x$. If $x$ is a point of $S = \partial B$ then $x$ is contained in (perhaps more than one) $(n - 1)$-simplex $\Delta_x$ of $T$. Let $\Delta'_x$ denote the face of $\Delta_x$ spanned by the vertices $v$ such that $t_i(v) = 1$ if $t_i(x) = 1$, and $t_i(v) = 0$ otherwise. (Note that if $x$ was on the boundary of $\Delta_x$, so that it was also contained in some other $(n - 1)$-simplex of $T$, then we still end up with the same simplex $\Delta'_x$ of $T$.) So, for example, if $x$ is at the barycenter of $\Delta_x$ then $\Delta'_x = \Delta_x$. By construction, for each vertex $v$ of $\Delta'_x$ the surface $H_x$ is pinched to a point along a disk $D$ in the isotopy class of $\iota(v)$. Hence, for all $y$ near $x$ the disk $D$ is a compression for $H_y$ that is disjoint from $F$. It follows that the entire simplex $\iota(\Delta'_x)$ is contained in $V_x$, and thus we may choose the barycenter of $\iota(\Delta'_x)$ to be the image of $\Psi(x)$. In particular, if $x$ is the barycenter of $\Delta_x$ then $\Psi(x) = \iota(x)$.

We now proceed to define the rest of the map $\Psi$ by induction. Let $\tau$ be a $d$-dimensional cell of $\Sigma$. By induction, assume $\Psi$ has been defined on the $(d - 1)$-skeleton of $\Sigma$. In particular, $\Psi$ has been defined on $\partial \tau$. Suppose $\sigma$ is a face of $\tau$. By Claim 3.3 $V_\sigma \subset V_\tau$. By assumption $\Psi|\sigma$ is defined and $\Psi(\sigma) \subset V_\tau$. We conclude $\Psi(\sigma) \subset V_\tau$ for all $\sigma \subset \partial \tau$, and thus

\begin{equation}
\Psi(\partial \tau) \subset V_\tau.
\end{equation}

Since $d = \dim(\tau)$ it follows from Claim 3.4 that $\pi_{(d-1)}(V_\tau) = 1$. Since $d - 1$ is the dimension of $\partial \tau$, we can thus extend $\Psi$ to a map from $\tau$ into $V_\tau$.

What remains to be shown is that if $\tau$ is in $S = \partial B$ then the extension of $\Psi$ from $\partial \tau$ to $\tau$ may be done so that $\Psi(\tau) \subset \iota(S)$. Let $\Delta_\tau$ be
Compressions for $H^F$ that are not compressions for $H$.

Compressions for $H$ contained in $M^F$.

**Figure 5.** Schematic showing how the complexes $\Gamma(H)$, $\Gamma_F(H)$, and $\Gamma(H^F)$ overlap.

By Equation 1, $\Psi(\partial \tau) \subset V_\tau \cap \iota(\Delta_\tau)$. Since $V_\tau \cap \iota(\Delta_\tau)$ will be a subsimplex of $\iota(\Delta_\tau)$, it follows that $\Psi$ can be extended over $\tau$ to this subsimplex.

By Equation 1, $\Psi(\partial \tau) \subset V_\tau$. So all we must do now is to show $\Psi(\partial \tau) \subset \iota(\Delta_\tau)$. Let $\sigma$ denote a face of $\tau$, and $\Delta_\sigma$ the simplex of $T$ whose interior contains $\sigma$. Then $\Delta_\sigma$ is contained in $\Delta_\tau$. By induction we may assume $\Psi(\sigma) \subset \iota(\Delta_\sigma)$. Putting this together we conclude $\Psi(\sigma) \subset \iota(\Delta_\tau)$ for each $\sigma \subset \partial \tau$, and thus $\Psi(\partial \tau) \subset \iota(\Delta_\tau)$. □

**Definition 3.5.** Let $F$ be a properly embedded surface in a 3-manifold $M$. Then we let $M^F$ denote the complement of a neighborhood of $F$ in $M$. For each subset $X$ of $M$, let $X^F = X \cap M^F$.

We define the complex $\Gamma(H^F)$ precisely as above, where the vertices of $\Gamma(H^F)$ correspond to the compressions for $H^F$ in $M^F$. The relationship between the complexes $\Gamma(H)$, $\Gamma(H^F)$, and $\Gamma_F(H)$ is depicted in Figure 5.

We now use Theorem 3.2 to show that when $H$ is topologically minimal and $F$ is incompressible, then $H$ may be isotoped so that $H^F$ is topologically minimal in $M^F$. In Section 4 we explore the implications of this when $H$ is a Heegaard surface.

**Lemma 3.6.** Let $M$ be an irreducible 3-manifold. Let $H$ and $F$ be properly embedded surfaces in $M$. Suppose $\Gamma_F(H)$ has well defined homotopy index. Then every loop of $H \cap F$ that is inessential on $F$ is inessential on $H$. 
Proof. Let \( \alpha \) denote a loop of \( H \cap F \) that is innermost among all loops that are inessential on \( F \), and essential on \( H \). As \( \alpha \) is inessential on \( F \), it cuts off a subdisk \( C' \) of \( F \). This disk can be pushed off of itself to be made disjoint from \( F \). Furthermore, any loop of \( C' \cap H \) that lies in the interior of \( C' \) must be inessential on \( H \). Thus, if \( \alpha \) bounds a compression for \( H \), then \( C' \) is a shadow for this compression. We claim \( \alpha \) does indeed bound such a compression.

If the interior of \( C' \) is disjoint from \( H \), then \( C' \) is a compression for \( H \), and we have established our initial claim. So suppose the interior of \( C' \) meets \( H \). Let \( \alpha^* \) denote a loop of \( C' \cap H \) that is innermost on \( C' \). Then \( \alpha^* \) bounds a subdisk \( C^* \) of \( C' \). By assumption, \( \alpha^* \) is inessential on \( H \), and thus it cuts off a disk \( D^* \) of \( H \). By the irreducibility of \( M \), the sphere \( C^* \cup D^* \) bounds a ball that we can use to guide an isotopy of \( C' \) that takes \( C^* \) to \( D^* \). This gets rid of one curve of \( C' \cap H \). Continuing in this way, we isotope \( C' \) to a disk \( C \) that it is a compression for \( H \).

We have thus shown \( C' \) is the shadow of a compression \( C \) for \( H \). It follows that \( C \in \Gamma_F(H) \), and thus \( \Gamma_F(H) \) is non-empty. Now suppose \( D \) is some other element of \( \Gamma_F(H) \), and let \( D' \) be the shadow of \( D \). Since \( D' \cap F = \emptyset \), \( \partial D' = \partial D \), it follows that \( \partial D \) is disjoint from \( F \). But since \( \alpha \subset F \), we conclude \( \partial D \) is disjoint from \( \alpha \). An innermost disk/outermost arc argument like the one given above then shows \( D \) and \( C \) can be made disjoint, and thus we conclude \( C \) is connected by an edge to every other element of \( \Gamma_F(H) \). It follows that \( \Gamma_F(H) \) is contractible to \( C \), a contradiction. \( \square \)

Lemma 3.7. Let \( F \) be a properly embedded, incompressible surface in an irreducible 3-manifold \( M \). Let \( H \) be a properly embedded surface in \( M \) such that \( \Gamma_F(H) \) has well defined homotopy index. Let \( D \) be a compression for \( H \) in \( M \) that is not a compression for \( H \). Then \( \Gamma_F(H/D) = \Gamma_F(H) \).

Proof. Let \( M(H) \) and \( B \) be as given in Definition 2.1. Then \( H/D \) is obtained from \( H \) by removing \( B \cap H \) from \( H \) and replacing it with the frontier \( D_* \) of \( B \) in \( M(H) \). As \( D \) is not a compression for \( H \), \( \partial D \) bounds a subdisk \( D \subset H \).

We first show \( \Gamma_F(H/D) \subset \Gamma_F(H) \). Suppose \( E \in \Gamma_F(H/D) \). Then \( \partial E \) can be isotoped off of \( D_* \). Hence, if \( E \) meets the ball \( B \) then it does so in loops of \( B \cap H \). A further isotopy makes \( E \cap H \) a collection of loops parallel on \( H \) to \( \partial D = \partial D \). But then each component of \( E \cap B \) can be swapped with a disk parallel to \( D \). The resulting disk has the same boundary as \( E \), but is disjoint from \( H \). By the irreducibility of \( M \) this disk must therefore be properly isotopic to \( E \). See Figure 6. We
Since $D$ is not a compression for $H$, any compression $E$ for $H/D$ (lower figure) is always isotopic to a compression for $H$ (upper figure). If $E'$ is a shadow for $E$ as a compression for $H/D$ (lower figure), then $E'$ is a shadow for $E$ as a compression for $H$ (upper figure). We now show $\Gamma_F(H) \subset \Gamma_F(H/D)$. Let $E$ now denote an element of $\Gamma_F(H)$. Thus, $\partial E \cap F = \emptyset$. If $\partial E$ meets $D$ then we may isotope it off, so that $\partial E \subset H/D$. This may introduce intersections of $E$ with $F$. But, as $F$ is incompressible, $E \cap F$ consists of loops that are inessential on $F$. Let $\beta$ denote a loop of $E \cap F$ that is innermost on $F$. Then $\beta$ bounds disks $C \subset E$ and $C' \subset F$. By Lemma 3.6, the disk $C'$ meets $H$ (and

conclude that $E$ is a compression for $H$ that persists as a compression for $H/D$. $E$ is therefore a compression for $H$ that is disjoint from $D$. Now let $E'$ be a shadow for $E$ as a compression for $H/D$. As $\partial E' = \partial E$, it follows that $\partial E' \cap D_* = \emptyset$. So, if $E'$ meets the ball $B$, then it meets it in disks parallel to $D$. The disk $E'$ thus meets $H$ in loops isotopic to $E' \cap H/D$, together with loops parallel to $D \cap H$. It follows that the interior of $E'$ meets $H$ in inessential loops, and thus, $E \in \Gamma_F(H)$. See Figure 6.
thus $H/D$ in inessential loops. By replacing $C$ with $C'$ we thus arrive at a disk with the same boundary as $E$, which meets $F$ fewer times, and whose interior meets $H/D$ in at most inessential loops. Continuing in this way we arrive at a disk $E'$ with the same boundary as $E$, which is disjoint from $F$ and whose interior meets $H/D$ in only inessential loops. The disk $E'$ is then a shadow for $E$, and thus $E \in \Gamma_F(H)$. □

**Theorem 3.8.** Let $F$ be a properly embedded, incompressible surface in an irreducible 3-manifold $M$. Let $H$ be a properly embedded surface in $M$ with topological index $n$. Then $H$ may be isotoped so that

1. $H$ meets $F$ in $p$ saddle tangencies, for some $p \leq n$. Away from these tangencies $H$ is transverse to $F$.
2. $H^F$ has topological index $i$, for some $i \leq n - p$.

**Proof.** We begin by isotoping $H$ so as to satisfy the conclusion of Theorem 3.2. Hence, we assume $H$ is tangent to $F$ in $p$ points, and the homotopy index of $\Gamma_F(H)$ is at most $n - p$.

Let $D$ be a compression for $H^F$ that is not a compression for $H$. Then $\partial D$ bounds a subdisk $D$ of $H$. By Lemma 3.7, $\Gamma_F(H/D) = \Gamma_F(H)$. The surface $H/D$ contains a component $H'$ isotopic to $H$ (by the irreducibility of $M$), and a surface isotopic to $D \cup D$. Note that as $D \cap F \neq \emptyset$, $H'$ meets $F$ fewer times than $H$ did. Thus, we may repeat the above procedure only finitely many times. Note also that this procedure will remove all center tangencies of $H$ with $F$. We arrive at a surface $H_*$ isotopic to $H$ with $\Gamma_F(H_*) = \Gamma_F(H)$, such that every compression for $H_*^F$ is also a compression for $H_*$. As such compressions lie in the complement of $F$, they are their own shadows. Hence, such compressions are elements of $\Gamma_F(H_*)$. We conclude $\Gamma(H_*^F) \subset \Gamma_F(H_*)$.

We claim the opposite inclusion is true as well, and thus $\Gamma(H_*^F) = \Gamma_F(H_*)$.

Suppose now $E \in \Gamma_F(H_*)$. Let $E'$ be a shadow of $E$. Let $\beta$ be a loop of $E' \cap H_*$ that is innermost on $H_*$. Then $\beta$ bounds subdisks $C \subset E'$ and $C' \subset H_*$. If $C' \cap F \neq \emptyset$, then $C'$ is a compression for $H_*^F$ that is not a compression for $H_*$, a contradiction. We conclude $C' \cap F = \emptyset$. It follows that we may replace $C$ with $C'$, transforming the disk $E'$ to a disk $E''$ such that $\partial E'' = \partial E$, $E'' \cap F = \emptyset$, and $|E'' \cap H_*| < |E' \cap H_*|$. Continuing in this way we arrive at a compression for $H_*$ with the same boundary as $E$, which is disjoint from $F$. Thus $E \in \Gamma(H_*^F)$.

We have now produced a surface $H_*$, isotopic to $H$, such that

$$\Gamma(H_*^F) = \Gamma_F(H_*) = \Gamma_F(H).$$

Thus, the homotopy index of $\Gamma(H_*^F)$ is equal to the homotopy index of $\Gamma_F(H)$. □
Corollary 3.9. Let $F$ be a properly embedded, incompressible surface in an irreducible 3-manifold $M$. Let $H$ be a properly embedded surface in $M$ with topological index $n$. Then $H$ may be isotoped so that it meets $F$ in a (possibly empty) collection of loops that are essential on both.

When $H$ is a Heegaard surface whose topological index is one this is a well-known result of Kobayashi that has been used extensively in the literature.

Proof. The first step is to use Theorem 3.8 to isotope $H$ so that $H^F$ is topologically minimal. The manifold $M^F$ is obtained from $M$ by removing a submanifold $N(F) \cong F \times I$. Let $F^1$ and $F^2$ denote the copies of $F$ on the boundary of $N(F)$. Each loop of $H \cap F^1$ is a loop or arc of $\partial H^F$. Hence, we must show that every loop of $\partial H^F$ that is inessential on $F^1$ is inessential on $H^F$. This is similar to Lemma 3.6.

If there is a loop of $\partial H^F$ that is inessential on $F^1$ then there is such a loop $\alpha$ that bounds a subdisk $C$ of $F^1$ whose interior is disjoint from $H^F$. If $\alpha$ is essential on $H^F$ then $C$ is a compression for $H^F$. Now suppose $D$ is some other element of $\Gamma(H^F)$. As $C \subset F^1$, the disks $C$ and $D$ can be made disjoint in $M$, and hence $(D,C)$ is an edge of $\Gamma(H^F)$. We conclude $C$ is connected by an edge to every other element of $\Gamma(H^F)$. It follows that $\Gamma(H^F)$ is contractible to $C$, a contradiction.

We conclude that all loops of $H \cap F^1$ that are inessential on $F^1$ are also inessential on $H$. Any such loop thus bounds a disk component of $H^F$ that can be isotoped into $N(F)$, without affecting $\Gamma(H^F)$. By successively performing this operation we thus arrive at the desired position of $H$ with respect to $F^1$, a surface isotopic to $F$. \hfill $\square$

4. Heegaard surfaces

In this section we give some applications of topological index theory to Heegaard splittings of 3-manifolds. We also show that the topological index of a surface is the sum of the topological indices of its components.

Lemma 4.1. Let $H$ be a properly embedded surface which separates $M$ into $\mathcal{V}$ and $\mathcal{W}$. Let $H_V$ be a surface obtained from $H$ by a sequence of compressions into $\mathcal{V}$. Then $H_V$ is incompressible in the submanifold cobounded by $H$ and $H_V$.

Proof. Let $\{D_i\}$ denote the union of the compressions used to obtain $H_V$ from $H$. Let $E$ denote a compression for $H_V$ that lies between $H$ and $H_V$. By an innermost disk argument, we may surger $E$ off of each disk $D_i$. But the complement of a neighborhood of $\bigcup D_i$ in this
submanifold is a product. As the boundary of a product does not admit compressions, we have thus reached a contradiction. □

**Lemma 4.2.** Let \( H \) be a properly embedded surface which separates \( M \) into \( V \) and \( W \). Let \( H_V \) and \( H_W \) be surfaces obtained from \( H \) by maximal sequences of compressions into \( V \) and \( W \). Let \( M_{VW} \) be the submanifold of \( M \) cobounded by \( H_V \) and \( H_W \). If \( H \) is topologically minimal in \( M \) then \( H \) is topologically minimal in \( M_{VW} \).

**Proof.** It suffices to show that every compression for \( H \) in \( M \) is isotopic to a compression in \( M_{VW} \). Let \( D \) be such a compression, and assume \( D \subset V \). Isotope \( D \) so that it meets \( H_V \) minimally. If \( D \cap H_V = \emptyset \), then the conclusion of the lemma follows. Hence, we assume there is a subdisk \( D' \) of \( D \), cut off by \( H_V \), whose interior is disjoint from \( H_V \). If \( D' \cap H_V \) is not essential, then we contradict our assumption that \( |D \cap H_V| \) is minimal. Hence, \( D' \cap H_V \) is essential and we conclude \( D' \) is a compression for \( H_V \).

If \( D' \) lies outside of \( M_{VW} \) then we contradict the maximality of the sequence of compressions used to obtain \( H_V \). But if \( D' \) lies in \( M_{VW} \) then it is in the submanifold cobounded by \( H \) and \( H_V \). This contradicts Lemma 4.1. □

**Theorem 4.3.** Let \( H \) be a properly embedded surface which separates \( M \) into \( V \) and \( W \). Let \( H_V \) be a surface obtained from \( H \) by a maximal sequence of compressions into \( V \). If \( H \) is topologically minimal then \( H_V \) is incompressible in \( M \).

**Proof.** Let \( H_W \) be the surface obtained from \( H \) by a maximal sequence of compressions into \( W \), and \( M_{VW} \) the submanifold of \( M \) cobounded by \( H_V \) and \( H_W \). By Lemma 4.2 the surface \( H \) is topologically minimal in \( M_{VW} \).

We now claim that if either \( H_V \) or \( H_W \) is compressible, then there is a compression for one that misses the other. Assume there is no such compression for \( H_W \). Let \( D \) be a compression for \( H_V \) in \( M \). Isotope \( D \) so that it meets \( H_W \) minimally. If \( D \) misses \( H_W \) then we establish our claim. Assume then that \( D \) meets \( H_W \). Let \( D' \) be a subdisk of \( D \) cut off by \( H_W \). If \( \partial D' \) is inessential on \( H_W \), then we contradict our assumption that \( |D \cap H_W| \) is minimal. But if \( \partial D' \) is essential on \( H_W \) then \( D' \) is a compression for \( H_W \) that misses \( H_V \), a contradiction. We conclude there is a compression \( D \) for either \( H_V \) or \( H_W \) that misses the other. That is, \( D \) is a compression for \( H_V \cup H_W \).

If \( D \) lies outside of \( M_{VW} \) then we contradict the minimality of the sequence of compressions used to obtain \( H_V \) or \( H_W \). Hence, \( D \subset M_{VW} \). Note that \( D \) is itself an incompressible surface. We may thus apply
Corollary 3.9 to isotope $H$ in $M_{ VW}$ to meet $D$ in a collection of loops that are essential on both surfaces. Since $D$ does not contain any essential loops, we conclude $D \cap H = \emptyset$.

The disk $D$ now lies either between $H$ and $H_V$, or between $H$ and $H_W$. In either case we contradict Lemma 4.1.

**Corollary 4.4.** Let $H$ be a topologically minimal Heegaard surface in a 3-manifold, $M$. Then $\partial M$ is incompressible.

In the topological index one case this follows also from a celebrated Lemma of Haken [Hak68]. In the topological index two case it was established by the author in [Bac08].

**Proof.** Let $V$, $W$, $H_V$, and $H_W$ be as in Theorem 4.3. Since $H$ is a Heegaard surface, every component of $\partial M$ is parallel to a component of either $H_V$ or $H_W$. The result is thus an immediate application of Theorem 4.3. □

**Corollary 4.5.** Let $H$ be a closed topologically minimal surface in an irreducible 3-manifold, $M$. Then either

1. $M$ contains a non-boundary parallel, incompressible surface,
2. $H$ is a Heegaard surface in $M$,
3. $H$ is contained in a ball, or
4. $H$ is isotopic into a neighborhood of $\partial M$.

In the next section we conjecture that the third possibility does not happen. In particular, if $M$ is a non-Haken 3-manifold then it would follow that every topologically minimal surface in $M$ is a Heegaard surface.

**Proof.** Let $V$, $W$, $H_V$, and $H_W$ be as in Theorem 4.3. Suppose first some component of $H_V \cup H_W$ is a sphere. By the irreducibility of $M$, this sphere bounds a ball. If the ball contains $H$, then the result follows. Otherwise, we may remove each such sphere component from $H_V \cup H_W$. If the resulting surfaces are boundary parallel, then either $H$ is contained in a neighborhood of some boundary component of $M$, or $H$ is a Heegaard splitting of $M$. If some component of $H_V \cup H_W$ is not boundary parallel then by Theorem 4.3 it is incompressible, and the result follows. □

**Lemma 4.6.** Suppose $F$ and $G$ are disjoint surfaces in an irreducible 3-manifold $M$, and $F \cup G$ is topologically minimal. Then $\Gamma(F \cup G)$ is the join of $\Gamma(F)$ and $\Gamma(G)$.

**Proof.** Let $H = F \cup G$. Let $V$, $W$, $H_V$, and $H_W$ be as in Theorem 4.3. By Theorem 4.3 the surfaces $H_V$ and $H_W$ are incompressible in $M$. 


If $E$ is a compression for $F$ then, as $H_V$ and $H_W$ are incompressible, we may isotope $E$ so that it is disjoint from both of these surfaces. It follows that $E$ is entirely contained in the component of $M_{VW}$ that contains $F$. But the surfaces $F$ and $G$ lie in different components of $M_{VW}$. Thus, $E$ must be disjoint from the surface $G$. Hence, any compression for $F$ is isotopic to a compression for $F \cup G$. We conclude there is a one-to-one correspondence between the vertices of $\Gamma(H)$ and the vertices of $\Gamma(F) \cup \Gamma(G)$. As every compression for $F$ will be disjoint from every compression for $G$, we conclude that $\Gamma(H)$ is the join of $\Gamma(F)$ and $\Gamma(G)$. □

**Theorem 4.7.** Suppose $F$ and $G$ are disjoint surfaces in an irreducible 3-manifold $M$, and $F \cup G$ is topologically minimal. Then $F$ and $G$ are topologically minimal and

$$\text{ind}(F) + \text{ind}(G) = \text{ind}(F \cup G).$$

Note that the hypothesis that $F \cup G$ is topologically minimal is extremely important. For example, let $F$ and $G$ be parallel surfaces in $M$ that each have topological index one. Then all of the compressing disks for $H = F \cup G$ are on the same “side” of $H$. Hence, by McCullough’s result [McC91], $\Gamma(H \cup G)$ is contractible. Thus $H$ does not have topological index two, as one might expect.

**Proof.** We first show that $F$ and $G$ are topologically minimal. If not, then $\Gamma(F)$ (say) is non-empty and contractible. But the join of a contractible space with any other space is also contractible. It thus follows from Corollary 4.6 that $F \cup G$ is not topologically minimal.

If either $F$ or $G$ has topological index 0 then the result is immediate. We assume, then, that the topological index of $F$ is $n \geq 1$ and the topological index of $G$ is $m \geq 1$.

By definition, $(n - 1)$ is the smallest $i$ such that $\pi_i(\Gamma(F)) \neq 1$, and $(m - 1)$ is the smallest $j$ such that $\pi_j(\Gamma(G)) \neq 1$. Our goal is to show that $(n + m - 1)$ is the smallest $k$ such that $\pi_k(\Gamma(F \cup G)) \neq 1$. By Corollary 4.6, this is equivalent to showing that $(n + m - 1)$ is the smallest $k$ such that $\pi_k(\Gamma(F) * \Gamma(G)) \neq 1$.

When $n = 2$ then $\pi_1(\Gamma(F)) \neq 1$. Suppose $F$ separates $M$ into $V$ and $W$. Let $\Gamma_V(F)$ and $\Gamma_W(F)$ denote the subsets of $\Gamma(F)$ spanned by the compressions that lie in $V$ and $W$, respectively. By an argument identical to the one given by McCullough in [McC91], $\Gamma_V(F)$ and $\Gamma_W(F)$ are contractible. If we contract these to points $p_V$ and $p_W$, then the remaining 1-simplices of $\Gamma(F)$ join these two points. The fundamental group $\pi_1(\Gamma(F))$ is generated by these 1-simplices. The remaining 2-simplices have become bigons that run once over each of two 1-simplices. Hence,
each such 2-simplex gives rise to a relation in $\pi_1(\Gamma(F))$ that kills one generator. It follows that $\pi_1(\Gamma(F))$ is free, and hence the non-triviality of $\pi_1(\Gamma(F))$ implies $H_1(\Gamma(F))$ is also non-trivial. Similarly, if $m = 2$ we conclude $H_1(\Gamma(G))$ is non-trivial. For $n \geq 3$ the non-triviality of $H_{n-1}(\Gamma(F))$ follows from the Hurewicz Theorem.

By Lemma 2.1 from [Mil68]:

$$\tilde{H}_{n+m-1}(\Gamma(F) \ast \Gamma(G)) \cong \sum_{i+j=n+m-2} \tilde{H}_i(\Gamma(F)) \otimes \tilde{H}_j(\Gamma(G)) + \sum_{i+j=n+m-3} \text{Tor}(\tilde{H}_i(\Gamma(F)), \tilde{H}_j(\Gamma(G))).$$

In particular, it follows from the fact that $(n-1)$ is the smallest $i$ such that $H_i(\Gamma(F))$ is non-trivial, and $(m-1)$ is the smallest $j$ such that $H_j(\Gamma(G))$ is non-trivial, that $(n+m-1)$ is the smallest $k$ such that $H_k(\Gamma(F) \ast \Gamma(G))$ is non-trivial. \qed

As an immediate corollary we obtain:

**Corollary 4.8.** If the topological index of $H$ is $n$, then the sum of the indices of the components of $H$ is exactly $n$. \qed

Combining Theorem 3.2 with Corollary 4.8 implies:

**Theorem 4.9.** Let $F$ be a properly embedded, incompressible surface in an irreducible 3-manifold $M$. Let $H$ be a properly embedded surface in $M$ with topological index $n$. Then $H$ may be isotoped so that

1. $H$ meets $F$ in $p$ saddles, for some $p \leq n$, and
2. the sum of the topological indices of the components of $H^F$, plus $p$, is at most $n$.

When $H$ is a Heegaard surface whose topological index is one, this result says that $F$ cuts $H$ up into incompressible pieces, along with at most one index 1 piece. Versions of this result were obtained by Schultens for graph manifolds [Sch04], and the author, Sedgwick, and Schleimer for more general Haken manifolds [BSS06].

Note also the similarity to the classification of almost normal surfaces given by Rubinstein. Such surfaces are cut up by the 2-skeleton of a triangulation into triangles and quadrilaterals, and exactly one “special” piece. Rubinstein [Rub95] and Stocking [Sto00] proved that topological index 1 surfaces can always be isotoped to be almost normal. The analogy is no coincidence. In the sequel [Baca] we show that when $H$ is topologically minimal and $K$ is the 1-skeleton, then
$H$ can be made topologically minimal, with respect to $K$. Combining this with the appropriate version of Theorem 4.9 then recovers the Rubinstein-Stocking result, and generalizes it to arbitrary topological index.

5. Questions

In any new theory, the questions raised are as important as the new results. Here we compile a list of questions and conjectures that we hope will stimulate further research on topologically minimal surfaces.

**Question 5.1.** How does topological index behave under finite covers? Are covers of topologically minimal surfaces also topologically minimal?

**Question 5.2.** Does every manifold have a topologically minimal Heegaard splitting?

**Question 5.3.** Are there non-Haken 3-manifolds with surfaces that have topological index $\geq 3$?

**Conjecture 5.4.** Suppose $M$ contains unstabilized Heegaard surfaces $F$ and $G$ that do not have topological index 1. Suppose further that the minimal genus common stabilization of $F$ and $G$ does not have topological index 2. Then $M$ contains a surface that has topological index 3.

By [CG87] such a manifold would be Haken, and so this conjecture compliments the question that precedes it.

**Question 5.5.** Is there a single 3-manifold that has surfaces of arbitrarily high topological index?

**Conjecture 5.6.** $S^3$ and $B^3$ do not contain topologically minimal surfaces.

A corollary would be that handlebodies do not contain closed topologically minimal surfaces. Note also that this conjecture rules out the third conclusion given by Corollary 4.5.

**Conjecture 5.7.** Let $F$ be a surface of positive genus. Then the only connected, topologically minimal surfaces in $F \times I$ are a single copy of $F$ and two copies of $F$ connected by an unknotted tube.

By the argument given in the proof of Corollary 4.5, any topologically minimal surface in $F \times I$ would be a Heegaard surface or would be contained in a ball. The only incompressible (i.e. index 0) surface in $F \times I$ is a copy of $F$. By [ST93], the only strongly irreducible (i.e. index 1) Heegaard surface is two copies of $F$ connected by an unknotted tube.
So, if Conjecture 5.6 is true, then Conjecture 5.7 is equivalent to the assertion that $F \times I$ contains no topologically minimal surfaces whose index is larger than one.

**Question 5.8.** Does the conclusion of Corollary 3.9 hold if $F$ is topologically minimal, but not incompressible?

Rubinstein and Scharlemann have shown [RS96] that Corollary 3.9 holds when $H$ and $F$ both have topological index 1. This was instrumental in their proof that there is an upper bound on the smallest genus of a common stabilization of Heegaard surfaces $F$ and $G$, in terms of the genera of $F$ and $G$.

**Conjecture 5.9.** If $H$ has topological index $n$ then it is isotopic to a geometrically minimal surface whose index is at most $n$.

The index 0 case was proved by Freedman, Hass and Scott [FHS83], and the index 1 case by Pitts and Rubinstein [PR87]. If true, it would indicate that topologically minimal surfaces are truly special. One would not expect, for example, a “random” surface in a 3-manifold to be isotopic to a minimal surface.

**Question 5.10.** Suppose $H$ has topological index $n$. What information does $\text{rank}(H_{n-1}(\Gamma(H)))$ carry? What about other algebraic invariants of $\Gamma(H)$?

**Question 5.11** (Generalized Hempel distance). For each surface $H$ there is a natural map of $\Gamma(H)$ into $\mathcal{C}(H)$, its curve complex, where the image of a compression $D$ is $\partial D$. By [Har86], $\mathcal{C}(H)$ has the homotopy type of a wedge of spheres. It follows that for low values of $n$ (in relation to the genus of $H$), each map $f : S^{n-1} \to \Gamma(H)$ can be extended to a map $\hat{f} : B^n \to \mathcal{C}(H)$. If we make all choices so that the number $d(n)$ of $n$-dimensional simplices in $\hat{f}(B^n)$ is minimal, then we get an interesting invariant when $f(S^{n-1})$ is not homotopic to a point in $\Gamma(H)$. When $H$ is a Heegaard surface that has topological index 1, Hempel called the invariant $d(1)$ the distance of $H$ [Hem01]. Many interesting results have been obtained about Hempel’s distance. What can be said about the invariant $d(n)$ for larger values of $n$?

**References**


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