HEEGAARD SPLITTINGS WITH BOUNDARY AND ALMOST NORMAL SURFACES (ADDENDUM)

DAVID BACHMAN

Abstract. The main content of this addendum is a more complete proof of Theorem 6.3 from [Bac01]. We also take this opportunity to correct a few minor errors that appeared in that paper.

We assume the reader is familiar with [Bac01]. The proof of Theorem 6.3 of that paper contains the phrase "Note that ∂B cannot compress away to nothing outside B, since M is not homeomorphic to S^3" (page 167, lines 4 and 5). While, strictly speaking, this statement is true it certainly requires further proof. This is the main content of this addendum.

One additional point which we address here is that Theorem 6.3 is stated only for 1-manifolds, but is used later for 1-vertex graphs. Furthermore, some of this graph may lie on ∂M, a point which was not emphasized in the original. The introductory paragraphs to Section 8 of [Bac01] were meant to deal with these technicalities, but some readers may find them somewhat unsatisfying. We will be much more precise with these issues here.

Before getting started the author would also like to apologize for not referencing the work of Yoav Reick and Eric Sedgwick [RS01] in [Bac01], since they had concurrently obtained many of the same results. The author is also grateful to Eric Sedgwick and Saul Schleimer for helpful conversations during the preparation of this note.

We begin by defining exactly what sort of graphs we will be dealing with.

Definition. A graph K embedded in a 3-manifold M is *locally untangled* if

(1) K has no vertices of valence zero, one, or two,
(2) the interior of each edge is either contained in ∂M or is disjoint from it, and
(3) for each ball $B \subset M$ with incompressible frontier in M^K there is a point $p \in K$ such that for all sufficiently small ϵ the frontier of B in M^K is parallel to the frontier of $N_\epsilon(p)$ in M^K.

It follows that a connected locally untangled graph has a single vertex. If not, then let e be an edge connecting distinct vertices. A neighborhood of e in M will be a ball violating the third condition of the definition.

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Examples of locally untangled graphs include 1-vertex spines of handlebodies bounded by strongly irreducible Heegaard splittings [Sch98] and 1-skeleta of *0-efficient* triangulations [JR]. It is the latter application that we make use of in [Bac01].

We now define the complexity of a surface with respect to a locally untangled graph. If F is a connected, properly embedded surface in M and K is a locally untangled graph then we define $c_K(F)$ to be $|F \cap K|$ plus

(1) 0 if F is a disk or sphere
(2) $1 - \chi(F)$ if F is closed
(3) $\frac{1}{2} - \chi(F)$ otherwise

If F is not connected then we define $c_K(F)$ to be $\sum_i c_K(F_i)$, where the sum ranges over all components F_i of F. *Maximal* and *minimal* leaves (with respect to K) of a singular foliation $\mathcal{F}$ are now defined in the obvious way, as well as the complexity $Lmax_K(\mathcal{F})$.

Definition. Let $\mathcal{F}$ be a singular foliation arising from a height function $h : M \to I$. A locally untangled graph K with vertex v is in *good position* if v is a local extremum of $h|K$. The graph K is *mini-Lmax with respect to* $\mathcal{F}$ if it is in good position, and K cannot be isotoped to reduce $Lmax_K(\mathcal{F})$.

If K is in good position, P^K is a maximal leaf of $\mathcal{F}^K$, and Q^K is the next minimal leaf then the region of M^K between P^K and Q^K is a ∂-compression body W^K. One ambiguity here is that for each ∂-compression body there are always many ways to assign ∂_+, ∂_-, and ∂_0. For the region between P^K and Q^K we will always make these assignments in the following way. If the vertex v of K is in W then let B be a neighborhood of v in M, and let S denote the frontier of B^K in M^K. Now, let

(1) $\partial_+ W^K = P^K$,
(2) $\partial_- W^K = Q^K \cup S$,
(3) and $\partial_0 W^K$ denote the remaining boundary of W.

It follows that $\partial_0 W^K \subset \partial N(K) \cup \partial_0 M$. If M is closed then these assignments are pictured schematically at the top of Figure 1. When M has boundary they are pictured at the bottom. Note that with these assignments a ∂_0-compression can run over maxima and minima of K, but not over the vertex.

One final correction is to the definition of a locally mini-Lmax foliation. On page 164, line 16 of [Bac01] we say, "we shall even refer to $\mathcal{F}$ as locally mini-Lmax if the maximal leaves are only quasi-strongly irreducible Heegaard surfaces." This sentence should be omitted. That is, a foliation is locally mini-Lmax only if it satisfies the conclusion of Theorem 5.2 of [Bac01].

The correct statement of Theorem 6.3 is as follows.

Theorem 6.3. *Let M be an irreducible 3-manifold other than S^3. Let K be a locally untangled graph in M with at least one edge whose interior is not on ∂M. Let $\mathcal{F}_*$ be a locally mini-Lmax foliation of M such that no loop component of K can be isotoped onto a leaf of $\mathcal{F}_*$. Then there is a locally mini-Lmax foliation $\mathcal{F}$ with the*

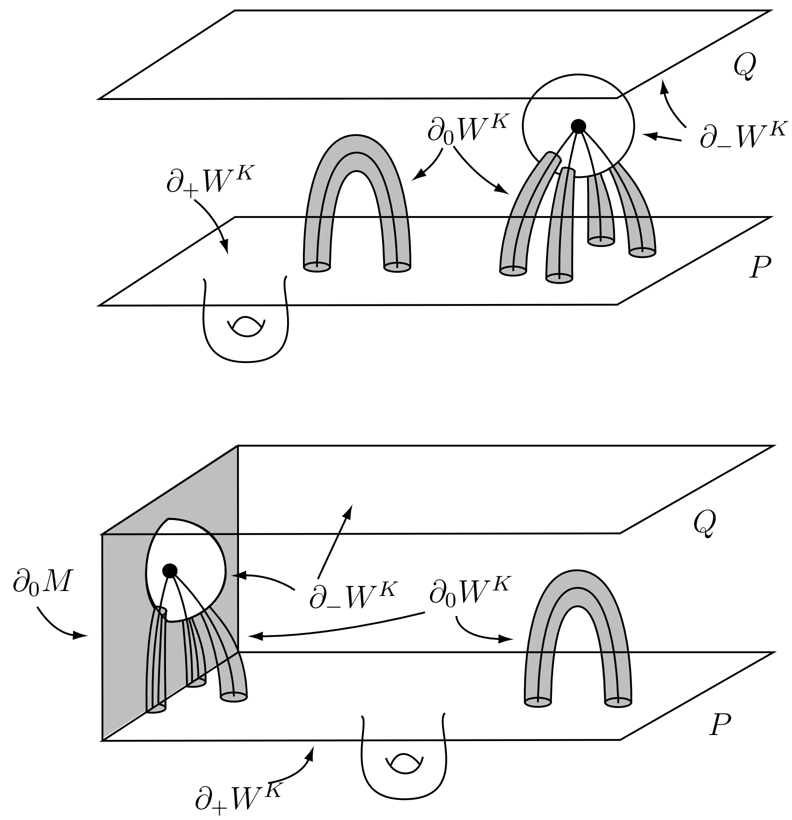

FIGURE 1

same maximal and minimal leaves as $\mathcal{F}_*$ (and hence the same Lmax complexity) such that when K is isotoped to be mini-Lmax with respect to $\mathcal{F}$ the foliation $\mathcal{F}^K$ has the following property. If K has a vertex then the maximal leaves of $\mathcal{F}^K$ are quasi-strongly irreducible ∂-Heegaard surfaces for the submanifolds of M^K that arise when we cut along the minimal leaves. If K has no vertex (i.e. it is a 1-manifold) then the maximal leaves are strongly irreducible, and hence $\mathcal{F}^K$ is a locally mini-Lmax foliation of M^K.

Before proceeding with the proof we will need the following lemmas.

Lemma 1. *Let L be a leaf of $\mathcal{F}$ and S be a sphere in M such that $S \cap K \subset L$ and $S \cap L$ is a connected surface. Then there exists a $K' \subset K$ such that $\partial K' \subset S \cap L$ and such that there is an isotopy of K which fixes $K \backslash K'$, and sends K' into $S \cap L$.*

Proof. Compress S as much as possible in M^K, to obtain a collection of spheres, R, in M. If any component of R bounds a ball which contains a single unknotted arc of K, then there is an innermost such, R'. The situation is now similar to the proof of the main result of [Tho97]. Inside R' there is a disk, T, such that $\partial T = \delta \cup \gamma$, where $\delta \subset \partial N(K)$ and $\gamma \subset R'$. We now reverse the compressions used to obtain R from

S. Each time a compression is reversed, we attach a tube to some components of R. These tubes may intersect T, but only in its interior. The arc δ is parallel in $N(K)$ to an arc $K' \subset K$. We can now use T to guide an isotopy of K which sends K' into S. As S is a sphere and $S \cap L$ is connected we may do a further isotopy of K' in S to bring K' into $S \cap L$.

Since K is locally untangled, the other possibility is that every component of R is parallel in M^K to the boundary of a neighborhood of the vertex of K. This case is similar to the main argument of [BS03], and the author is greatly indebted to Saul Schleimer for his collaboration on that work.

To fix notation, let $\{D_i\}_{i=1}^n$ be a sequence of disks, and $\{S_i\}_{i=0}^n$ the sequence of surfaces, so that $S_0 = S$, $S_n = R$, and S_i is obtained from S_{i-1} by compressing along D_i in the complement of K. That is, remove a small neighborhood, A_i, of ∂D_i from S_{i-1}. Construct S_i by gluing two parallel copies of D_i onto ∂A_i. Denote these by B_i and C_i. So $A_n \cup B_n \cup C_n$ bounds a ball homeomorphic to $D_n \times I$. Finally, let α be the image of $\{pt\} \times I \subset D_n \times I$ by such a homeomorphism. The arc α is a *co-core* for the compression D_n.

Let V be the component of S_{n-1} which meets D_n. Let $\{R_j\}_{j=0}^n$ denote the components of $R = S_n$, numbered consecutively, so that R_0 is farthest from the vertex of K. Then there is a j such that compressing V along D_n yields R_j and R_{j+1}. Let N denote the submanifold of M bounded by R_j and R_{j+1}. Note $\alpha \subset N$.

Choose a homeomorphism $h : S^2 \times I \to N$ such that $K \cap N$ is a collection of *straight arcs* and $\pi(h^{-1}(B_n)) = \pi(h^{-1}(C_n))$. Here a straight arc is one of the form $h(\{pt\} \times I)$ and $\pi : S^2 \times I \to S^2$ is projection onto the first factor.

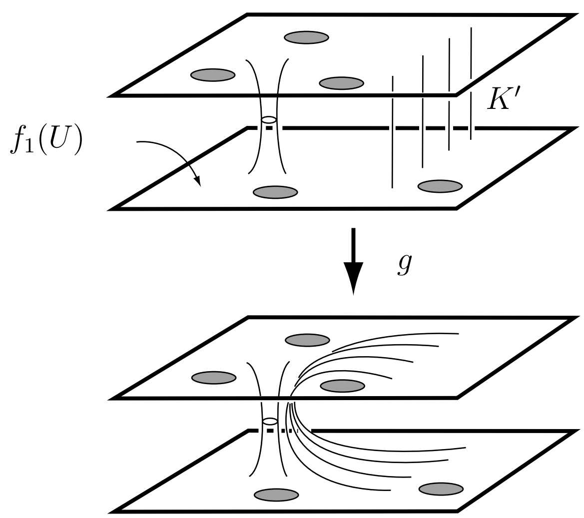

FIGURE 2

The usual *lightbulb trick* (see [Rol76], for example) implies that there is an isotopy, $f : M \times I \to M$, fixing the complement of N pointwise such that $f_1(\alpha)$ is a straight arc (where $f_t(x)$ is shorthand for $f(x,t)$). Let $U = V \cap S$ $(= V \setminus (\bigcup_{i=0}^{n-1} B_i \cup C_i))$. Let $K' = K \cap N$. Let $g : M \times I \to M$ be an isotopy which fixes the complement of N pointwise, such that $g_1(K') \subset f_1(U)$. Such an isotopy exists since the arcs of K' are straight, and since $f_1(U)$ follows the boundary of a neighborhood of $f_1(\alpha)$, which is also straight (see Figure 2.) Note that $f_1^{-1}\big(g_1(K')\big) \subset U \subset S$. As before, we may now do a further isotopy of K' in S to bring K' into $S \cap L$. □

Lemma 2. *Let L be a leaf of $\mathcal{F}$ and S be a disk properly embedded in M such that*

1. *∂S bounds a disk on $\partial_0 M$ which contains the vertex of K,*
2. *$S \cap K \subset L$, and*
3. *$S \cap L$ is a connected surface.*

Let B be the ball in M bounded by the disk S and some subdisk of $\partial_0 M$. Then either

- *there exists a $K' \subset K$ such that $\partial K' \subset S \cap L$ and such that there is an isotopy of K which fixes $K \backslash K'$, and sends K' into $S \cap L$ or*
- *all of the arcs of $K \cap B$ which do not lie on $\partial_0 M$ are simultaneously parallel to arcs of $K \cap B$ which do lie on $\partial_0 M$.*

Proof. Compress and ∂_0-compress S as much as possible in M^K to obtain a collection of disks and spheres, R, in M. If any component of R is a sphere which bounds a ball containing a single unknotted arc of K, then there is an innermost such, R'. Inside R' there is a disk, T, such that $\partial T = K' \cup \theta$, where K' is a subarc of K and $\theta \subset R'$. We now reverse the compressions used to obtain R from S. Each time a compression is reversed, we attach a tube to some components of R. These tubes may intersect T, but only in its interior. We can now use T to guide an isotopy of K which sends K' into S. As $\partial K' \subset S \cap L$ and $S \cap L$ is a connected subsurface of a disk we may do a further isotopy of K' in S to bring K' into $S \cap L$.

Since, if there are any sphere components of R we are done, we now assume that only ∂-compressions were done to obtain R from S. As K is locally untangled we conclude that R is made up of possibly several trivial disks (*i.e.* disks which cobound a ball, with a disk in $\partial_0 M$, containing an unknotted arc of K) and at least one copy of a vertex-linking disk (*i.e.* a disk which is parallel to the frontier of a neighborhood of the vertex of K). Note that there must be at least one vertex-linking disk component of R. (Otherwise S would meet K only in points on the boundary of M. This, in turn, implies that all of the edges of K are on ∂M, which is not the case.)

Let R'' denote a vertex-linking component of R which is closest to the vertex. Every component of $K \cap B$ whose interior is in the interior of M either now lies inside R'', or between two vertex-linking components of R. In either case each such subarc is parallel to a subarc of K on $\partial_0 M$. Furthermore, undoing all of the ∂-compressions used to obtain R from S does not disturb this parallelism. □

We now proceed with the proof of Theorem 6.3. Choose a locally mini-Lmax foliation $\mathcal{F}$ with the same minimal and maximal leaves as $\mathcal{F}_*$ so that when K is isotoped to be mini-Lmax with respect to $\mathcal{F}$ the complexity $Lmax_K$ is minimal. Let P be a leaf of $\mathcal{F}$ such that P^K is a maximal leaf of $\mathcal{F}^K$. If P^K is the first maximal leaf then let W be the region of M below P. Otherwise, let Q be a leaf of $\mathcal{F}$ such that Q^K is the minimal leaf of $\mathcal{F}^K$ which comes just before P^K, and let W be the region of M between Q and P. Similarly, if P^K is the last maximal leaf then let W_* be the region of M above P. Otherwise, let Q_* be a leaf of $\mathcal{F}$ such that Q_*^K is the minimal leaf of $\mathcal{F}^K$ which comes just after P^K, and let W_* be the region of M between P and Q_*.

Claim 3. *The submanifolds W and W_* are irreducible.*

Proof. The only situation in which W is not a ∂-compression body (and hence not irreducible) is when some component of Q is a sphere. We now invoke Lemma 1 (where $S = Q$) to isotope some subset of K into Q, while keeping the rest fixed. Pushing the interior of this subset just out of W shows that Q can not be a minimal leaf of $\mathcal{F}$. □

By way of contradiction, assume that there are compressing or honest ∂_0 compressing disks $D \subset W^K$ and $D_* \subset W_*^K$ for P^K such that $D \cap D^* = \emptyset$. Note that W^K is a ∂-compression body and $\partial_0 W^K \subset \partial_0 M \cup \partial N(K)$.

Definition. An honest ∂_0-compressing disk for P^K in W^K or W_*^K is a *low disk* or *high disk* (respectively) if $\partial D = \alpha \cup \beta$, where $\beta \subset \partial N(K)$.

Definition. A compressing or honest ∂_0-compressing disk for P^K in W^K or W_*^K is *real* if it is a compressing or ∂_0-compressing disk for P. Otherwise it is *fake*.

Given this definition, we may now list all of the cases that will have to be considered (up to symmetry).

(1) D and D_* are real.
(2) D and D_* are fake.
(3) D is fake and D_* is real.

<u>Case 1.</u> *The disks D and D_* are real.*

Then not only is P^K a maximal leaf for $\mathcal{F}^K$, but also P is a maximal leaf of $\mathcal{F}$. If $\partial D \cap \partial D_* = \emptyset$ then P fails to be a strongly irreducible ∂-Heegaard surface, and hence, $\mathcal{F}$ is not locally mini-Lmax. Since the local mini-Lmaximality of $\mathcal{F}$ was a hypothesis of the Theorem we have reached a contradiction. This concludes Case 1.

Following Case 1 we surmise that at least one of D and D_* is fake. Hence, before proceeding to the remaining cases we are forced to analyze all types of fake disks. Assume D is fake.

(1) D is a compressing disk for P^K. Then $\partial D = \alpha$ is an essential curve on P^K, but is inessential on P. Hence, α bounds a disk $E \subset P$ which is punctured by K. As W is irreducible (by Claim 3), $D \cup E$ bounds a ball $B \subset W$.
   (a) B contains the vertex of K. Note that this can only happen when M is closed. We will call such a disk a *vertex compression*.
   (b) B does not contain the vertex. Then B contains an arc of K which is parallel into P. Hence, there is a low disk in B. In this case the fake disk D will be referred to as a *cap*. A low disk in B will be referred to as being *inside* the cap D.

(2) D is an honest ∂_0-compressing disk for P^K. Then $\partial D = \alpha \cup \beta$, where $\alpha \subset P^K$ is essential on P^K and $\beta \subset \partial_0 W^K$.
   (a) $\beta \subset \partial N(K)$. Then D is a low disk.
   (b) $\beta \subset \partial_0 M$. Then there is a disk $E \subset P$ punctured by K such that $\partial E = \alpha \cup \gamma$, where $\gamma \subset \partial P$. If β was essential on $\partial_0 W$ then $D \cup E$ would be a compressing disk for $\partial_0 W$, which is impossible. (To see the contradiction, double W along $\partial_0 W$. The compressing disk $D \cup E$ then turns into an essential 2-sphere in a compression body.) We conclude β is inessential on $\partial_0 W$, and hence cobounds a disk V. If $V \cap K = \emptyset$ then D would not be an honest ∂_0-compressing disk for P^K. Hence, V either contains a low disk or it contains the vertex of K. In the former case the disk D will be referred to as a *∂-cap*. In the latter it will be referred to as a *vertex ∂-compression*. If D is a ∂-cap then a low disk in V is said to be *inside* D.

Case 2. *D and D_* are fake.*

This case is organized into subcases as follows:

(2.1) D and D_* are compressing disks.
(2.2) D_* is a compressing disk and D is a ∂-compressing disk.
(2.3) D and D_* are ∂-compressing disks.

Subcase 2.1. If D and D_* are compressing disks then ∂D bounds a disk E in P and ∂D_* bounds a disk E_* in P. As $D \cap D_* = \emptyset$ the following is a complete list of the subcases that will have to be considered, up to symmetry.

(2.1.1) $E \cap E_* = \emptyset$
(2.1.2) $E_* \subset E$

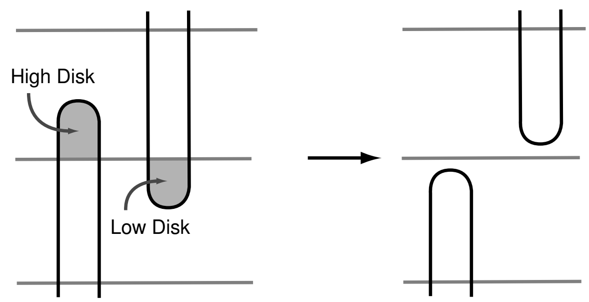

Figure 3

Subcase 2.1.1. $E \cap E_* = \emptyset$. If D and D_* are caps then there is a low disk inside D which is disjoint from a high disk inside D_*. We can then do the move depicted in Figure 3 to reduce $Lmax_K(\mathcal{F})$. On the other hand, if one of D or D_* is a vertex compression then we can perform the move depicted in Figure 4 to lower $Lmax_K(\mathcal{F})$.

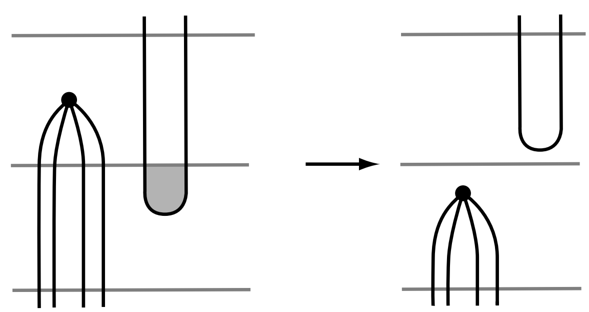

Figure 4

Subcase 2.1.2. $E_* \subset E$. Then ∂D and ∂D_* cobound an annulus $A \subset P$. We assume that D_* is outermost, in the sense that there does not exist a compressing disk $D'_* \subset W_*$ for P^K such that $\partial D'_* \subset A$, but is not parallel to ∂D_* in P^K. Let S denote the sphere $D \cup A \cup D_*$. We now claim that S^K is incompressible in M^K. Suppose not.

Invoke Lemma 1 to isotope some subset K' of K into A. If the interior of K' met at least one maximal leaf of $\mathcal{F}^K$ then this isotopy produces a foliation with smaller $Lmax_K$. Furthermore, since we have only done an isotopy of K we have not disturbed the foliation $\mathcal{F}$, so we have reached a contradiction.

If the interior of K' does not meet any maximal leaves of $\mathcal{F}^K$ then it also must not meet any minimal leaves. Hence, every arc of K' lies in W or W_*. If some such arc, γ, were in W then it would also be in the ball bounded by $D \cup E$. As it is isotopic to an arc in A there is a low disk $L \subset W$ such $L \cap P \subset A$. But then this low disk is disjoint from E_*, and we may do one of the moves depicted in Figure 3 or 4, according to whether D_* is a cap or a vertex compression.

We conclude $\gamma \subset W_*$. As it is isotopic to an arc in A there is a high disk $H \subset W_*$ such $H \cap P \subset A$. Let H' be a cap in W_* whose boundary is in A, such that the high disk H is inside H'. Let s be any arc in A^K joining $\partial H'$ to ∂D_*. Then there is a component of the frontier of a neighborhood of $H' \cup s \cup D_*$ in W_* which is a cap which is "more outermost" than D_*, contradicting our assumption.

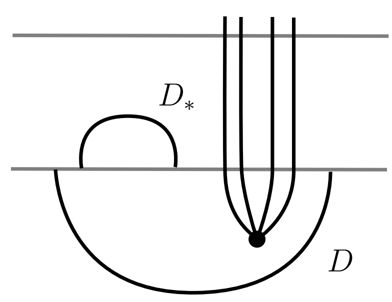

FIGURE 5

We are now faced with the possibility that S^K was incompressible to begin with. As the only such spheres are parallel to the vertex of K we may now isotope K to look like Figure 5 in the ball bounded by S. As this isotopy necessarily reduced the number of intersections of K with P, the complexity $Lmax_K$ has gone down.

Subcase 2.2. If D_* is a compressing disk and D is a ∂-compressing disk then ∂D_* bounds a disk $E_* \subset P$ and $\partial D = \alpha \cup \beta$ where $\alpha \subset P$. There are now three further subcases.

(2.2.1) D is a ∂-cap
(2.2.2) D is a low disk
(2.2.3) D is a vertex ∂-compression

Subcase 2.2.1 If D is a ∂-cap then there is a low disk D' inside D which is disjoint from E_*. If D_* is a cap then there is a high disk inside D_*, which is then disjoint from D'. We can now do the *Lmax* lowering move depicted in Figure 3. Similarly, if D_* is a vertex compression then we can do the move depicted in Figure 4.

Subcase 2.2.2 If D is a low disk then there is a cap D' such that D is inside D' and $D' \cap D = \emptyset$. This case now reduces to Subcase 2.1.

Subcase 2.2.3 The disk D is a vertex ∂-compression, and hence $\beta \subset \partial_0 W$. It follows that D_* is a cap, since the vertex of K is in W. The curves ∂D and ∂D_* cut off subdisks E and E_* of P.

If $E \cap E_* = \emptyset$ then there is a high disk inside D_* which we can use to perform the $Lmax_K$ lowering move depicted in Figure 4. Hence, $E_* \subset E$, and we are in a situation similar to Subcase 2.1.2 (see Figure 6 left). Let A be the closure of $E - E_*$. We assume that D_* is outermost, in the sense that there does not exist a compressing disk $D'_* \subset W_*$ for P^K such that $\partial D'_* \subset A$, but is not parallel to ∂D_* in P^K. Let S denote the disk $D \cup A \cup D_*$.

By Lemma 2 there are two possibilities. The first is that some subarc K' of K is isotopic into A. If the interior of K' met at least one maximal leaf of $\mathcal{F}^K$ then this isotopy produces a foliation with smaller $Lmax_K$. Furthermore, since we have only done an isotopy of K we have not disturbed the foliation $\mathcal{F}$, so we have reached a contradiction.

If the interior of K' does not meet any maximal leaves of $\mathcal{F}^K$ then it also must not meet any minimal leaves. Hence, K' lies in W or W_*. If K' were in W then it would also be in the ball bounded by $D \cup E \cup V$. As it is isotopic to an arc in A there is a low disk $L \subset W$ such $L \cap P \subset A$. But then this low disk is disjoint from E_*, and we may do the move depicted in Figure 3.

We conclude $K' \subset W_*$. As it is isotopic to an arc in A there is a high disk $H \subset W_*$ such $H \cap P \subset A$. Let H' be a cap in W_* whose boundary is in A, such that the high disk H is inside H'. Let s be any arc in A^K joining $\partial H'$ to ∂D_*. Then there is a component of the frontier of a neighborhood of $H' \cup s \cup D_*$ in W_* which is a cap which is "more outermost" than D_*, contradicting our assumption.

The second possibility implied by the conclusion of Lemma 2 is that the subarcs K'' of K which join the vertex of K to A, and whose interiors lie in the interior of M, are isotopic to subarcs of K on $\partial_0 M$ (see Figure 6 middle). If any of the arcs of K'' met E_* then this isotopy lowers the complexity $Lmax_K$. If not then we see there is a fake compressing disk, D', whose boundary is disjoint from D_* (see Figure 6 right), and we are reduced to Subcase 2.1.2.

Subcase 2.3. If D and D_* are ∂-compressing disks then the following table lists all subcases, up to symmetry.

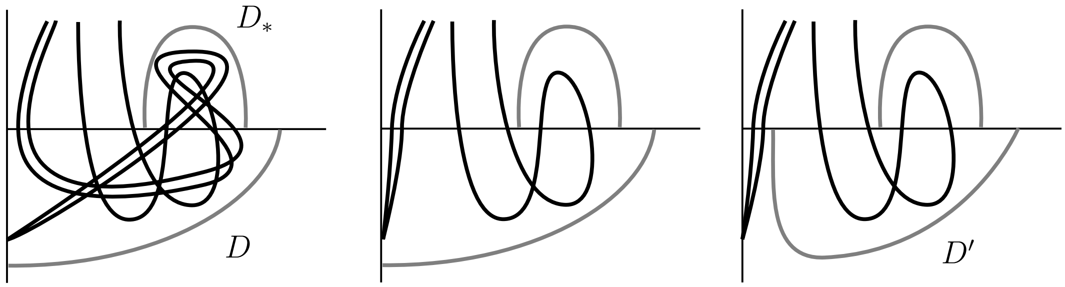

FIGURE 6

$D \backslash D_*$	∂-cap	high disk	vertex ∂-compression
∂-cap	(2.3.1)	(2.3.1)	(2.3.3)
low disk		(2.3.1)	(2.3.2)
vertex ∂-compression			Not possible

Subcase 2.3.1. If D is a low disk or a ∂-cap and D_* is a high disk or a ∂-cap then there is a low and a high disk that we can use to perform one of the moves depicted in Figures 3 or 7.

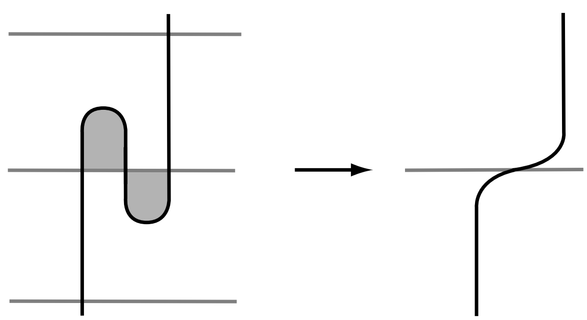

FIGURE 7

Subcase 2.3.2. If D is a low disk in the interior of M and D_* is a vertex ∂-compression then there is a cap D' in W such that D is inside D', and $\partial D' \cap \partial D_* = \emptyset$. We are now reduced to Subcase 2.2.3. If $D \subset \partial_0 M$ then there is a ∂-cap D' which contains D such that $\partial D' \cap \partial D_* = \emptyset$. This is handled in Subcase 2.3.4.

Subcase 2.3.3. The final case is when D_* is a vertex ∂-compression and D is a ∂-cap. Then ∂D and ∂D_* cut off subdisks E and E_* of P. Inside D there is a low disk. If $E \cap E_* = \emptyset$ then we may use this low disk to perform the move depicted in Figure 4. We conclude $E_* \subset E$ or $E \subset E_*$. Let S denote the disk comprised of E, E_* and the subset of P which lies between E and E_*.

Just as in Subcase 2.2.3 we now invoke Lemma 2 with the disk S. As before the first possibility is that some subarc K' of K is isotopic into A. We rule this possibility out by reasoning which is identical to that in Subcase 2.2.3.

Let B denote the ball bounded by S and a subdisk of $\partial_0 M$. The second possibility given by Lemma 2 is that all of the arcs K'' of $K \cap B$ which do not lie on $\partial_0 M$ are isotopic to subarcs of K which do lie on $\partial_0 M$. But we can always isotope the arcs on $\partial_0 M$ to look like one of the two possibilities given in Figure 8 (depending on whether $E \subset E_*$ or $E_* \subset E$). Dragging the arcs K'' along with such an isotopy reduces $Lmax_K$. Note that in the figure on the left the vertex of K has crossed P.

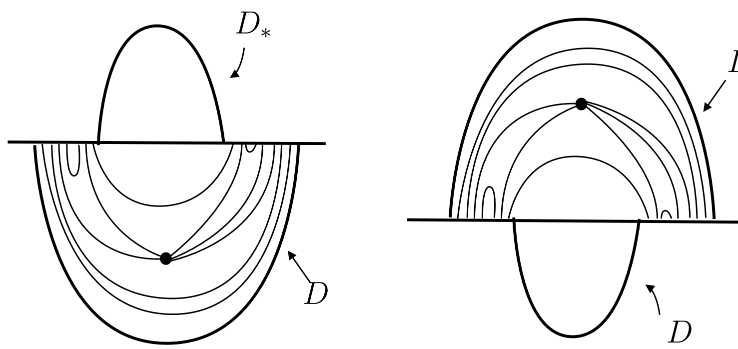

FIGURE 8

<u>Case 3.</u> *D is fake and D_* is real.*

In W_* we alter the foliation $\mathcal{F}$ as follows. Note that Q_* is obtained from P by a sequence of compressions. These compressions are realized by the foliation $\mathcal{F}$ in the sense that as t (the height parameter for $\mathcal{F}$) increases we see the leaves compress in some sequence. But note that any sequence of compressions that one can perform on P to obtain Q_* can be realized by some foliation in this way. Choose some such sequence which begins with a compression along the disk D_*. Alter the foliation in W_* so that the leaves compress along this sequence. Note that the maximal and minimal leaves of $\mathcal{F}^K$ are unchanged, so the complexity $Lmax_K$ is unchanged as well.

The disk D_* now represents a compression or ∂_0-compression of the leaves of $\mathcal{F}$. If D is a low disk then we may use it to isotope a subarc of K so that we pass through the corresponding minimum of K *after* we see the compression/∂_0-compression which corresponds to D_*. Such a move reduces the complexity $Lmax_K$, a contradiction.

If D is a compression then we let B be the ball bounded by D and some subdisk of P. If D is a ∂-compression then let B be the ball bounded by D, a subdisk of P, and a subdisk of $\partial_0 W$. In either case we can use B to define an isotopy of K so that any minimum or vertex of K which is inside B gets pushed past the compression/∂_0-compression which corresponds to D_*. Again, such a move reduces the complexity $Lmax_K$.

In summary, we have shown that if D is any compressing (or honest ∂_0-compressing) disk for P^K in W^K, and D_* is a compressing (or honest ∂_0-compressing) disk for P^K in W_*^K, then $\partial D \cap \partial D_* \neq \emptyset$. Hence, P^K is a quasi-strongly irreducible ∂-Heegaard surface for $(W^K) \cup (W_*^K)$. Furthermore, the only times the assumption of honesty was used was in cases where K had a vertex. Hence, if K has no vertex then we conclude P^K is strongly irreducible.

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Mathematics Department, California Polytechnic State University

E-mail address: dbachman@calpoly.edu