Entity versus Process Conceptions of Error Bounds in Students' Reinvention of Limit Definitions

Robert Raish
University of Northern Colorado
robert.raish@unco.edu

Michael Oehrtman
University of Northern Colorado
Michael.Oehrtman@unco.edu

Jason Martin
University of Central Arkansas
jasonm@uca.edu

Brian Fisher
Pepperdine University
brain.fisher@pepperdine.edu

Craig Swinyard
University of Portland
swinyard@up.edu

We report results from a guided reinvention of the definition of sequence convergence conducted in three second-semester calculus classes. This report contributes to the growing body of research on how students come to understand and reason with formal limit definitions, focusing on the emergence of students' understanding of the epsilon quantity, conceived in terms of error bounds. Using Sfard's framework of the condensation of processes to entities, we mapped the possible conceptual trajectories followed by the students in the study. In this report, we detail our map, these trajectories and students' reasoning about other aspects of the formal definition, and the influence of reasoning about approximations and error analyses in students' progression.

Key words: error bound, formal limit definition, guided reinvention, sequence convergence

Introduction

Limit concepts are central to the study of calculus, but students often develop only a weak understanding of limits that does not support their learning or reasoning about other central concepts in calculus (Davis, 1986; Oehrtman, 2009; Sierpinska, 1987; Tall, 1980; Tall, 1981). Recently, a series of guided reinventions where pairs of students reinvented formal definition for concepts related to limits have provided multiple insights into how students come to understand limits (Hart-Weber et al., 2011; Martin & Oehrtman, 2010; Martin et al., 2011; Martin et al., 2012; Oehrtman et al., 2011; Swinyard, 2011). We employed techniques of the guided reinvention heuristic from Realistic Mathematics Education (Gravemeijer, 1998) in three second-semester calculus classrooms to engage students in reinventing the formal definition of sequence convergence, typically expressed as[1],

> A sequence (a_n) converges to L provided that for every $\varepsilon > 0$, there exists an N such that $| a_n - L | < \varepsilon$ whenever $n \geqslant N$.

We posed the following research questions: 1) What cognitive challenges do students encounter while reinventing the formal definition of sequence convergence? 2) How do students resolve these cognitive challenges? 3) To what extent and in what ways do the students use approximations and error analyses in their reinvention? 4) How does a guided reinvention of the definition of sequence convergence in a

[1] The definition commonly stated in this form presumes without explicit statement that ε is quantified as a real number while N and n are quantified as natural numbers.

classroom differ from a guided reinvention with a pair of students? This research report addresses the first three questions focusing on the students' emerging understanding of the quantity typically represented by ε in the formal definition.

Literature Review

Early literature on how students understand limits focused on informal reasoning, conceptual barriers, and misconceptions (e.g., Bezuidenhout, 2001; Cornu, 1991; Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Monaghan, 1991; Tall, 1992; Williams, 1991). Cottrill et al. developed a genetic decomposition outlining a sequence of 7 mental constructions that students might make when coming to understand limit concepts (1996). This decomposition was one of the first attempts to understand how students reason with formal definitions. Unfortunately he was only able to find evidence support the first four mental constructions and was not able to collect any data on the last three.

Building on research by Williams (1991) and Tall and Vinner (1981) Oehrtman (2009) identified 5 different metaphors students use when attempting to reason with limits: "collapse in dimension", "approximation", "closeness in spatial domain", "physical limitations," and "infinity as a number." Oehrtman (2008) found that focused instruction could systematize students' use of approximation metaphors enabling more powerful subsequent reasoning about how limit structures were manifest in the various concepts developed throughout introductory calculus. When one teaches reinforcing the systematic exploration of approximation and error analyses reflecting limit structures in calculus, we will say that one is teaching using the approximation framework. Oehrtman (2008) also hypothesized that such a structural approach to calculus could establish a strong conceptual foundation for later formalization of limit concepts, such as creating and reasoning with rigorous definitions.

Very little existing research provides insight into *how students do come to understand the formal definition*. Multiple teaching experiments have been performed in the last few years to help better understand how students do come to understand the formal definition by having pairs of students reinvent the formal definition of convergence of a sequence of numbers (Hart-Weber et. al., 2011; Martin, Oehrtman 2010, Martin et. al., 2011; Martin et. al. 2012, Oehrtman et.al., 2011; Swinyard & Larsen, 2012). These teaching experiments begin with students constructing a rich set of examples of sequences that converge to a limit L and examples of sequences that do not converge to L serving an explicit external representation of their concept image of convergence of a sequence. Students then engage in an iterative process in which they are asked to create a definition, evaluate the definition against examples and non-examples, identify conflicts with their set of examples, then attempt to resolve these conflicts (Oehrtman et al., 2011). Oehrtman et al. identified *problems* and *problematic issues* that students encountered when reinventing formal limit convergence. *Problems* are cognitive challenges that students explicit identified as causing conflict between their concept image and their stated definition. *Problematic issues* are cognitive challenges that students do not produce cognitive conflict in their concept image, but are in conflict with the formal definition.

One of the more complex aspects of students developing an understanding of the formal definition appears to be their conception of the quantity ε and its relation to other elements in the definition. Swinyard and Larsen have proposed that an important aspect to coming to understanding the formal definition is that students develop a *y-first perspective*. When students are reasoning informally with they traditionally start with an *x-first perspective*, but in the formal definition we consider a bound on the range of y-values first and then consider values on the x-axis (Swinyard & Larsen, 2012). They also noted that the first four step of the genetic decomposition involve an x-first perspective.

This research proposes a model of when students adopt a y-first perspective what are possible trajectories that they could possibly take by focusing on error bounds. We provide a classification of different ways that students learn to reason with error bounds and the possible progression that students may take when reinventing the formal definition of sequence convergence. Our final model of how students how students reason with error bounds is framed in terms of Sfard's theoretical characterization of the condensation of processes into an entity. Sfard defines a process to be manipulation of familiar objects and an entity to be a condensed process. Sfard classifies any *process* as an operational conception, which is understood to not involve static objects. If an individual posses an entity view then one possess a structural conception, which involves a n individual thinking of an abstract object. Condensation of a process into an entity is said to have occurred if one is able to simultaneously consider a collection of process without having to consider an individual action (Sfard, 1992). Oehrtman, Swinyard, and Martin (in preparation) suggest that the condensation of the process of determining an N value based on an ε value may be a crucial step in constructing a universal quantification on ε in the definition.

Methodology

The research was conducted in three Calculus 2 classes with different instructors at two medium-sized research universities midway through the spring semester. The three Calculus 2 classes and the Calculus 1 classes taken by nearly all of the students the previous fall were taught using activities developed with Oehrtman's (2008) approximation framework. Since these activities were implemented in weekly in-class labs requiring students to collaborate in small groups, all of the students were familiar with this format. During the guided reinvention we engaged students in five class sessions working in groups of four or five to construct a definition of sequence convergence based on a collection of examples and non-examples the classes had previously generated. Each group had a large whiteboard used for writing out definitions and communicating ideas. One group in each class was selected to be video and audio recorded at all times, and a second video camera in each class captured the whole-class interactions. All documents created by the students were also collected.

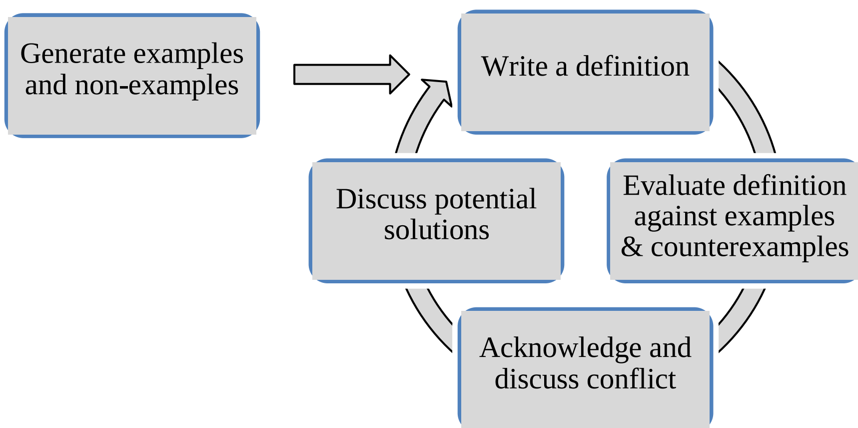

Figure 1. Iterative refinement in the process of guided reinvention of a formal definition.

Throughout the reinvention students completed reflections on the examples and nonexamples, their emerging definition, and problems they identified with their definition. For the first reflection, which occurred prior to the guided reinvention, students were asked to create graphs of as many qualitatively distinct examples of sequences that converge to five and sequences that do not converge to five as possible. The researchers compiled representative samples of these graphs on a handout for students to reference throughout the guided reinvention. On the first day of the reinvention, all groups were given the prompt "A sequence converges to 5 provided that ..." The facilitators then guided students through the iterative process of writing a definition, evaluating their definition against the examples and nonexamples, identifying problems, and proposing solutions (Figure 1).

There were 6-7 groups in each class and two undergraduate facilitators in addition to the instructor and one member of the researcher team supporting the groups. The facilitators' role included keeping students engaged in the iterative refinement cycle, asking clarifying questions, and introducing cognitive dissonance when students were not aware of critical problems with their definitions.

After each class session the research team would meet to discuss interesting interactions that occurred in class and modify the protocol for subsequent days. Members of the research team would then watch the videos from each group before the next class meeting. This report focuses on our emerging hypotheses concerning how students conceived of and reasoned with the ε quantity in the formal definition and its interpretation in terms of an error bound as developed in the approximation framework. We evaluated these hypotheses against classroom events during each research team meeting and rewrote our hypotheses between each session. While reviewing the individual group video, we then developed predictions for how these ideas would develop during the next session. After the completion of the guided reinvention, we created content logs while again reviewing the video data, coding statements about the ε quantity in the students' definition. We then refined our hypotheses until we felt we could adequately understand all of the students' statements about ε and error bounds and could explain the challenges they encountered while reasoning about this quantity and the shifts that the made when reasoning with error bounds.

Results

Our initial hypothesis about students' reasoning about the ε quantity can be summarized as follows:

> Students will eventually frame their definition in terms of approximating the value of the limit L with terms of the sequence a_n and errors $| a_n - L |$. This formulation will still reflect students' initial intuitive domain-first images of terms approaching the limit and thus will not incorporate aspects of an error bound. As students encounter problems with their definition applied to specific examples (such as a damped oscillation around L) and nonexamples (such as a sequence monotonically increasing to a value slightly smaller than L), they will recognize a need to say how close a sequence needs to get to its limit. In conjunction with their language and notation about approximations, this recognition will trigger a recall and application of ideas about error bounds. Students' initial attempts will involve only a single value for ε, but as they recognize a need to rule out every possible nonexample, they will eventually construct a universal quantification for ε.

Our initial analysis agreed with our initial hypothesis that students do tend to begin with a domain-first perspective, but had difficulty explaining the difficulties that students had with developing a universal quantification of error bounds. Framing students' reasoning with Sfard's framework of condensation of processes into entities we were able to differentiate the ways that students think about error bounds. Once students begin to think of error bounds there appears to be two common trajectories that students can take. These two trajectories appear to be a reflection of the two types of questions in the approximation framework activities related to error bounds. One type of question asks students to calculate error bounds from information about the approximations. When referring to this type of conception of error bound we will use the notation E.B. When calculating error bounds in the activities a common strategy is to find an overestimate and an underestimate and use the difference between the two as an error bound. The other type of question gives students an error bound as a predetermined tolerance and then asks them to find a subset of the domain that produces approximations within that tolerance. As part of this question there is usually a follow-up question asking if it is possible to find a subset of the domain that produces approximations with error less than any tolerance one may want. When referring to such a tolerance conception of error bounds, we will use the notation ε.

The two trajectories that students usually take involve students' first thinking of error bounds as a process. One way that a student may possess a process view of error bounds is related to the type of question that ask students to calculate an error bound. A process view of error bounds related to this question would result in student manipulating objects or information about these objects to construct an error bound or a sequence of error bounds. One of the ways that students constructed error bounds is by finding an overestimate and an underestimate and subtract them, as is typically expect in the approximation framework. From these constructed error bounds students then usually will talk about the error bounds "getting smaller," "approaching zero," "decreasing," or other similar phrases. Students were not necessarily computing numerical values, but it appears that they are constructing error bounds from properties the approximations. For example in the first explicit discussion on error bounds of the participants said “the error bound between these two points like this and two points like this is going to be so close to the same thing.” Here we can see that this individual is using the points to understand the error bound and that the error bound is derived from information about the points. It appears that students may condense E.B.'s as an entity view of error bounds. To think of E.B.'s as an entity one must be able to construct a collection of error bounds simultaneously without having to consider any single action. Data seems to indicate that E.B. entity manifests themselves as "trend lines" of the error bounds that the approximations must lie within.

Another process view of error bound is related to the second question where students are asked to find approximations within a given tolerance, ε. Here students consider a single error bound and then find a relationship of the *n*-value(s) and/or the approximation(s) to this error bound. For example on the second day one of the participant’s definition included “that as n increases beyond ten thousand all of the values of a sub n gets so close to five that the error becomes less than this number here.” It should be noted that possessing this entity view of error bounds does not mean that an individual has a complete understanding of the formal definition of sequence convergence. For example one of the groups articulated that for any error bound they are able to find a single approximation with error less than any error.

Conclusion

Our data analysis identified two distinct trajectories for the development of students’ conceptions of error bounds (see Figure 2). The students in our study first developed a process view of terms of the sequence approaching the limit, which was eventually framed in terms of approximations then augmented to include associated errors. Some students then condensed their process into an entity view of the errors, enabling them to reason about an entire collection of approximations and their errors simultaneously. At this point, the students typically described errors decreasing in holistic terms such as “past some point n, $|a_{n+1}-5|<|a_n-5|$” or described the errors for an entire set of approximations to be small, such as “past here, all of the errors are negligible.” . Regardless of whether students developed an entity view of errors, all eventually began to think of one of the two conceptions of error bound characterized in our results. The E.B. view does not easily lead to the universal quantification of ε which is needed to understand the formal definition of sequence convergence. In fact, all of the students we observed using an E.B. view eventually backtracked and explicitly started thinking in term so of the ε quantity as a tolerance before conceiving of a universal quantification.

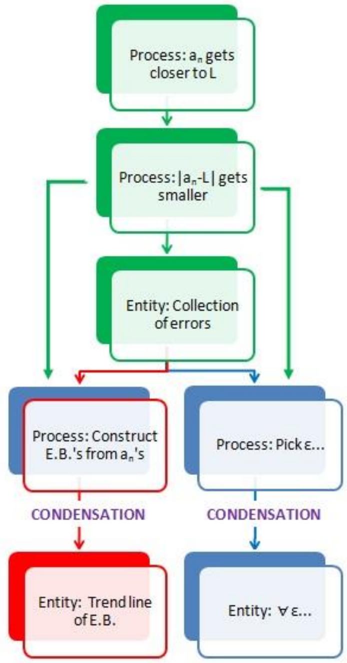

Figure 2. Trajectory of error bound conceptions

We hypothesize that develop an ability to fluidly move between E.B. and ε views of error bounds is crucial students learning to reason about formal limit definitions. In subsequent work, we are analyzing the interactions of students various process and entity conceptions of the ε quantity in formal limit definitions with the other elements of the definition.

References

Artigue, M. (1991). Analysis. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 167–198). Dordrecht, The Netherlands: Kluwer.

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science & Technology, 32*(4), 487-500.

Cornu, B. (1991). Limits. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Davis, R. B., Vinner, S. (1986) The notation of limit: Some seemingly unavoidable misconception stages. The Journal of Mathematical Behavior, 5, 281-303.

Hart-Weber, C., Oehrtman, M., Martin, J., Swinyard, C., & Roh, K. (2011). *The nature and effect of idiosyncratic examples in student reasoning about limits of sequences.* In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the Fourteenth Conference on Research in Undergraduate Mathematics Education* (Vol. 4, pp. 87-91), Portland, OR: Portland State University.

Martin, J., & Oehrtman, M. (2010). Strong metaphors for the concept of convergence of Taylor series. Paper presented at the 13th Annual Conference on Research in Undergraduate Mathematics Education, Raleigh, NC: North Carolina State University.

Martin, J., Oehrtman, M., Roh, C., Swinyard, C., & Hart-Weber, C. (2011). *Students' Reinvention of Definitions of Series and Pointwise Convergence*. Paper accepted to the Fourteenth Conference on Research in Undergraduate Mathematics Education, Portland, OR.

Martin, J., Oehrtman, M., Roh, C., Swinyard, C., & Hart-Weber, C. (2011). Students' reinvention of definitions of series and pointwise convergence. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the Fourteenth Conference on Research in Undergraduate Mathematics Education* (Vol. 1, pp. 239-254), Portland, OR: Portland State University.

Martin, J., Cory, B., Oehrtman, M., & Swinyard, C. (2012). Reinvention Six Months Later: The Case of Megan. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the Fifteenth Conference on Research in Undergraduate Mathematics Education* (pp. 109-115), Portland, OR: Portland State University.

Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics, 11*(3), 20-24.

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (Vol. 73, pp. 65-80). Washington, DC: Mathematcial Association of America.

Oehrtman, M. (2009). Collapsing dimensions, physical limitations, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, 40(4), 396-426.

Oehrtman, M., Swinyard, C., Martin, J., Roh, K., & Hart-Weber, C. (2011). From intuition to rigor: Calculus students' reinvention of the definition of sequence convergence. Paper presented at the 14th Annual Conference on Research in Undergraduate Mathematics Education, Portland, OR: Portland State University.

Oehrtman, M., Swinyard, C., & Martin, J., (in preparation). Problems and solutions in students' reinvention of a definition for sequence convergence.

Sfard, A. (1992). Operational origins of mathematical objects and the quandry of reification—the case of function. In G. Harel & E. Dubinsky (eds.), The Concept of Function: Aspects of Epistemology and Pedagogy, MAA Notes 25, (pp.59-84). Washington DC: MAA.

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics, 18,* 371–397.

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy & Mike. *Journal of Mathematical Behavior, 30*, 93-114.

Swinyard, C., & Larsen, S. (2012). What does it mean to understand the formal definition of limit?: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education, 43,* 4, 465-493.

Tall, D. (1980). Mathematical intuition, with special reference to limiting processes. Proceedings of the Fourth International Conference for the Psychology of Mathematics Education, 170-176.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education, 22*(3), 219-236.

Vygotsky, L. (1987). The development of scientific concepts in childhood. In R. W. Rieber & A. S. Carton (Eds.), *Problems of general psychology: Vol. 1. The collected works of L. S. Vygotsky* (pp. 167–241). New York: Plenum. (Original work published 1934)