A Modern Look at the Cell Problem

Jennifer A. Czocher, The Ohio State University

A models & modeling theoretical perspective has been suggested to supersede both constructvism (Lesh, Doerr, Carmona, & Hjalmarson, 2003) and problem-solving paradigms (Zawojewski, 2007). This is important to the RUME community because many of our foundational works are rooted in theoretical and methodological perspectives derived from these paradigms. In the case of problem solving and modeling, it is important to re-view the problem-solving research settings with a mathematical modeling lens. Without this glance backwards we cannot connect new ideas to old knowledge and we should not supplant a theory without ensuring the next can account for existing observations. The objective of this paper is to revisit a well-known problem setting to explore alternative interpretations of students' mathematical work.

The Task This report is part of a larger study to examine how students mathematically structure nonmathematical settings. The task considered is one of 17 modeling tasks used in the larger study which were designed to elicit the cognitive and mathematical activities attendant to mathematical modeling. The Cell Problem was selected for closer inspection because of its connection to and impact on the course of mathematics education in the US (see Schoenfeld, 1982a): *Estimate how many cells there are in the average adult human body. (How much faith do you have in this figure? What about a lower estimate? An upper estimate?)* The problem fits the criteria of a Fermi problem, a type of estimation problem championed for its ability to require the modeler to identify conditions, assumptions, relevant variables and parameters, and to estimate values for those parameters (Sriraman & Lesh, 2006; Ärlebäck, 2009). The Cell Problem provides an ideal setting to re-examine interpretation of students' problem-solving activity because the task lends itself to several readily realized mental models (e.g., arising from weight and density, partitioning, percentages).

Previous Findings The Cell Problem is well-known for its use in RUME in the early 80's (Schoenfeld, 1982a). These students sought increasingly finer estimates of the volume of the human body, and in other studies (e.g., Schoenfeld, 1982b, 1985) they carried on with such "wild goose chases" without pausing to evaluate their productivity. Schoenfeld concluded that students' metacognitive control was failing. Schoenfeld (1982b) further argued that even in methodologically "clean" laboratory settings, subjects' responses may be influenced by (i) a need to produce *something* due to the pressure o being recorded (ii) the expectation that some methods are "more legitimate" for solving problems than others, and (iii) the solver's own beliefs about the nature of mathematics. These observations and suggestions all point to why a student may have failed to produce certain expected mathematical behaviors and they contribute substantially to our understanding of how mathematical thinking is socially constituted in an interview setting. However, some aspects – such as the students' interpretation of the problem's context – are missing from the picture. From a models & modeling perspective, this aspect is central to interpretation of the student's behavior.

Research Questions The questions guiding analysis were: (i) How did the students make sense of the problem context? (ii) How did the students examine their own productivity?

Mathematical Modeling Theory A *model* is a simplified representation of some system. *Modeling* refers to both a sequence of behaviors and a way of thinking about a problem (Kehle & Lester, 2003). These behaviors include a proclivity to describe, explain, or interpret phenomena in mathematical terms (Lesh & Yoon, 2007). A mathematical model is the ordered triple (S, M, R), where S is a situation in the real world, M is its mathematical representation, and R is an invertible, idiosyncratic, cognitive link between the two (Blum & Niss, 1991). The modeler's success is a function of the information that hte modeler takes into account, how he accesses and harnesses conceptual models, choices in symbolization, and use of symbolic intuition (see Shternberg & Yerushalmy, 2003) to attribute meaning to the model. Mathematical modeling is theorized as a cyclical, iterative process that connects the Real World to the Mathematical World and can be seen in Figure 1. According to the theory, a modeler's activity can be resolved into stages (labeled by letters [a] - [f]) and transitions among those stages (labeled by numbers [1]-[6]). The stages and transitions are in Figure 1, Tables 1 & 2.

Methods For the larger study, four engineering undergraduates enrolled in differential equations were selected among volunteers to participate in a series of seven one-on-one, task-based cognitive interviews focused on mathematical modeling. Each interview session lasted an hour to an hour-and-a-half, with the Cell Problem taking between 16 and 23 minutes and demonstrating the full scope of the modeling cycle. The interview sessions were semi-structured and I interacted with students to ask follow up questions, to clarify my understanding of their statements, or to challenge their assertions. I also encouraged the students to use resources they felt might help them (e.g., textbooks, internet searches, calculators). Video recordings of the sessions were reviewed and transcripts of the sessions were segmented into statements containing a complete idea and these were tagged, via the method of constant comparison, with externally observable indicators corresponding to cognitive activities in the modeling cycle (activities [1]-[6]) (Borromeo-Ferri, 2007). The research questions were operationalized in terms of Figure 1: *understanding* and *simplifying/structuring* activities were associated with sense-making and *validating* with the students' verification activities. Instances of these activities were examined for technique of validation, the factors in S that were selected to become represented in M, and an emergent theme centered on the students' perceived importance of *accuracy and precision.*

Interview Results and Discussion The Cell Problem data is comprised of protocols from four male engineering undergraduates: Orys, Trystane, Mance, and Torrhen. The students spent extended periods of time in the *simplifying/structuring* and *validating* transitions, as is typical of a Fermi-type problem. Findings are organized thematically.

Sense-making. In this task, all students selected measurable parameters (eg, average human volume, size of the average cell) and all students were able to see multiple ways of structuring the problem multiplicatively, either as a weight or volume model. In contrast to Schoenfeld students, these students spent more time worrying about the impact of cell size than human size. Trystane and Orys were both concerned over cell size a function of its type. Since Orys found an estimate for one dimension of the average cell, he assumed the cells were spherical and viewed the task as a packing problem. Trystane solved it as a partitioning problem, breaking up the body in to different types of cells. Trystane iterated interpretations of the problem as changes in parameter sets. Using google, he adjusted M to reflect the parameters he had data for. In contrast, Mance's activity revealed a negative relationship among *simplifying/structuring*, *mathematization*, and *validation*: he created three situation models (based on weight, surface area, and volume) but could not disentangle one relevant variable set from the others during *validation*, leading to *competing conceptual systems* (see Lesh & Yoon, 2007). In terms of Figure 1, Mance was unable to produce a multiplicative mathematical representation M for any of the variable sets because the relationship *density* $\times$ *volume* = *weight* persisted each time he tried to build R.

Validating The students primarily validated their models using empirical and experiential comparisons to their real results (model predictions) and also dimensional analysis. Orys and Torrhen commented that it was difficult to validate the results because the numbers were so large "that they are just *big* and the meaning is lost in the physical sense even though it still has meaning in the mathematical sense." Other than comparing the prediction to an empirical value or number sense, Trystane compared his mathematical model (a weighted average) against a real model (a diagram of a human divided into cell types according to organ or system). This kind of validating was not predicted by the modeling cycle.

Accuracy and Precision Torrhen was the only student who gave a "ballpark" estimate. Mance dismissed results of such models as being inaccurate and so not worthwhile estimates. Orys voiced his concern over his spherically-shaped cells and about averaging over cell types. Trystane lamented that the average of so may kinds of cells was not useful. He concluded, "I don't know the context of the question. Why would someone ask this?...I can't see why having the number of cells in the entire body would be useful." Thus, without an idea of what the answer was going to be used *for* (eg, a measure of health), it was impossible for him to determine error tolerance. Taken together these students' responses highlight a few issues in validating mathematical models: accuracy of the parameter estimates (and how errors compound), the representativeness of *average* and whether it can be substituted for a set of measurements, accuracy of the prediction based on these assumptions, and the precision needed to answer the question (determined by the *purpose* for posing the question).

Mance's case revealed that the struggle to articulate a mathematical model may be hindered by multiple competing situation models. These participants did not behave pathologically, but rather spent their time seeking information that had meaning according to their situation and real models, and then adjusting their models in light of available information. These students were wary of small variations in cell volume and its impact on cell count. Schoenfeld's students may have resorted to adjusting human parameters because they did not have information available about cell sizes, because human dimensions are more familiar to the human senses, or because humans can be replaced with sets of familiar shapes. The students in this sample tended to rely on empirical and experiential methods of validation.

Mathematical modeling theory predicts that both identification of relevant variables and relationships (*simplifying/structuring*) and *validating* activity critically affect model construction. However, in the case of The Cell Problem, aspects of the *sensibility* of the task were not considered, and these are important from a modeler's perspective: Who would want to know the answer? For what purpose? The students were left to guess answers to these questions, and therefore were unable to interpret the problem as intended. Thus, in order to observe this kind of mental activity in students' mathematical work, we must provide problem contexts that are commensurate with their means of carrying it out. This work is ongoing, but it is sufficient to show that mathematical modeling perspectives can provide for full and sensible reinterpretations of existing observations.

Questions for the Panel How can I explore the role of information requests and uses in mathematical modeling? Is *purpose* of the modeling task as important to accuracy when there is no numerical result? Are there other foundational works about which our understanding might benefit from re-visits with a mathematical modeling lens?

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Figure 1. Mathematical modeling cycle (Blum, 2011)

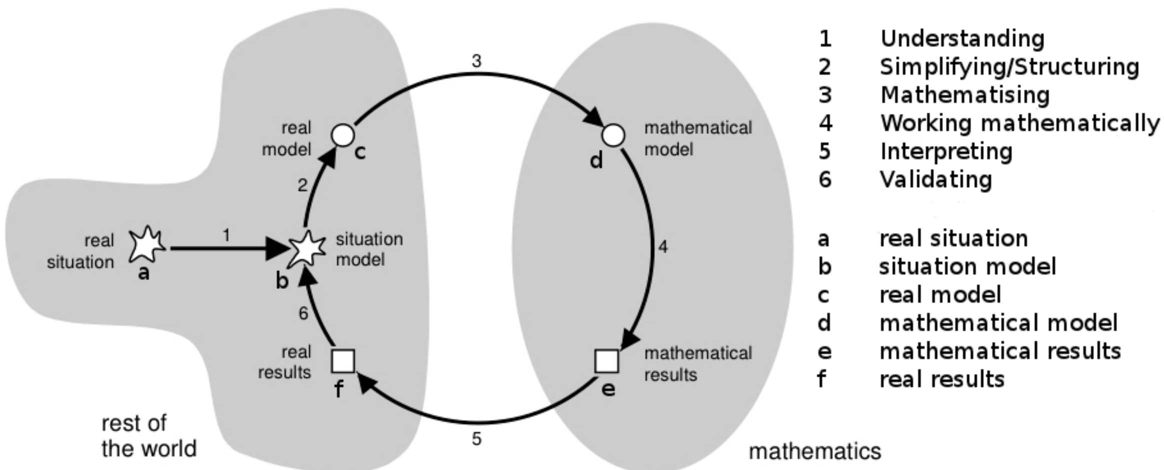

Table 1: Stages of Mathematical modeling

Stage	**Description**
real situation	problem, as it exists
situation model	conceptual model of problem
real model	idealized version of the problem (serves as basis for mathematization)
mathematical model	model in mathematical terms
mathematical results	results of mathematical problem
real result	answer to real problem

Table 2: Cognitive Activities of Mathematical Modeling

Activity	**Trying to Capture**
understanding	forming an idea about what the problem is asking for
simplifying/structuring	identify critical components of the mathematical model (ie, create an idealized view of the problem)
mathematizing	represent the real model mathematically
working mathematically	mathematical analysis
interpreting	recontextualizing the mathematical result
validating	verifying results against constraints