MATHEMATICIANS' TOOL USE IN PROOF CONSTRUCTION

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This study sought to describe the tools and reasoning techniques used by mathematicians to construct and write proofs. Task-based clinical interviews were conducted with 3 research mathematicians in varying research fields. The tasks were upper-undergraduate and lower-graduate level proofs from linear algebra, basic analysis, and abstract algebra. Data were coded based on a framework constructed from Dewey's theory of inquiry and the characterizations of conceptual insight and technical handle. Preliminary results indicate the task of discovering a conceptual insight that can potentially lead to a proof can be problematic, and there are distinct moments in the construction process when the problem changes from "why should this be true?" to "how can I prove that?"

Key Words: Proof, Inquiry, Experts

Harel and Sowder (2007) called for a comprehensive perspective on proof; they stated that a central goal of teaching proof is to "gradually help students develop an understanding of proof that is consistent with that shared and practiced by the mathematicians of today. (p. 47)" Consequently, it is necessary to develop an accurate and comprehensive understanding of mathematicians' proof construction processes and the reasoning techniques by which they construct proof. This paper reports preliminary results from research intended to describe expert mathematicians' inquiry into proof construction. We specifically analyze experts' inquiry in regards to developing and implementing a conceptual insight and technical handle (Raman, Sandefur, Birky, & Somers, 2009).

Background/Theoretical Perspective

We apply Dewey's theory of inquiry and tool-use (Dewey, 1938; Hickman, 1990) to analyze mathematicians' proof construction process. Dewey defines inquiry as the intentional process to resolve doubtful situations, through the systematic invention, development, and deployment of tools (Hickman, 2011). A tool is a theory, proposal, or knowledge chosen to be applied to a problematic situation. Throughout the entire inquiry process, there is an "end-in-view" (Garrison, 2009; Glassman, 2001; Hickman, 2009). These ends-in-view provide tentative consequences which the inquirer must seek the means (tools and ways to apply tools) to attain. These ends-in-view may be modified and adapted as the inquiry process proceeds.

The process of active, productive inquiry involves reflection, action, and evaluation. Reflection is indeed the dominant trait. The inquirer must inspect the situation, choose a tool to apply to the situation, and think through a course of action. After this initial reflection of what could happen, the inquirer performs an action, applies the tool. In these actions, the inquirer operates in some way on the situation; she applies a tool to the situation, thus altering it. Reciprocally, during or after the fulfilling experience, the inquirer evaluates the effects and appropriateness of the application of the chosen tools (Hickman, 1990).

We attempt to focus on understanding the mechanisms that lead to new insights. Raman and colleagues (Raman, Sandefur, Birky, Campbell, & Somers, 2009; Raman & Weber, 2006) have developed a model for describing student difficulties for proof production, including the moments of finding a conceptual insight (sometimes termed key idea) and a technical handle. Attaining a conceptual insight gives the prover a sense of conviction and why a particular claim is true. A technical handle is an idea that renders the proof communicable; discovering a technical handle gives the prover a sense of "now I can prove it" (Raman, et al. 2009). These constructs characterize moments when the prover creates a new insight, an instance of the invention and deployment of a tool. We believe that focusing on these specific moments distinguishes this work from others' research which focuses on rich, broad descriptions of the process of problem solving (i.e. Carlson & Bloom, 2005).

This report details preliminary findings for research into the following questions: (1) What tools and reasoning techniques are used by mathematicians in search of a conceptual insight and technical handles? (2) How do professional mathematicians use conceptual insights and technical handles as tools in constructing and writing proof?

Methods

The participants of the study consisted of 3 professional mathematicians. Each participant engaged in a task-based interview that included three proof construction tasks and follow-up questions. The tasks were upper-undergraduate and lower-graduate level proofs from linear algebra, basic analysis, and abstract algebra. Participants were video recorded, their work was recorded using a LiveScribe pen, observation notes were taken by the pair of interviewers, and interviews were transcribed. We developed a coding scheme based on Dewey's theory of inquiry (Dewey, 1938; Hickman, 1990) and Raman and colleagues' characterization of conceptual insight and technical handles (Raman, et al., 2009; Raman & Weber, 2006). In applying the coding scheme, we parsed transcripts into "major events," or individual actions or groups of actions involving one purpose or one problem. We coded each major event by type of experience, problem, tool-used, purpose of tools-used, type of evaluation and mode of thinking. We described problems and tools in context for clarity. We then further subdivided major events if we determined that more than one purpose or more than one problem occurred in its duration. Coders added additional codes for problems, tools, and purpose of tools as needed.

Results

From our coding, it was apparent that the participants began each problem first attempting to manipulate the premises they were given in order to get a sense of the mathematics they were engaging. Then, participants generally began applying tools with the purpose of looking for conceptual insight, or a sense of belief and insight into the reason why the statement is true.

After reading the analysis task, *[Let f be a continuous function defined on I=[a,b], f maps I onto I, f is one-to-one, and f is its own inverse. Show that except for one possibility, f must be monotonically decreasing on I.]*, Dr. K stated, "What I'm puzzled by is why it has to be decreasing." We interpret this as articulation of a problem of not seeing a conceptual insight. Dr. K applied the tool of turning "it into a geometry problem" by drawing a picture. He based his diagram on his conceptual knowledge of what it would mean for a continuous function to be one-to-one and onto: "it can't go up and down" and for a function to be its own inverse: "it has to be symmetric when I flip it over the line." He deemed his picture as fitting both the hypothesis and the conclusion, but he still wanted "to think about why that's true." He then drew the identity function and noted that it was the one exception. Finally he drew another picture including the line $y = x$ and a point located above the line $y = x$. He applied his previously developed knowledge of the unique exception and his conception of the graph needing to be symmetric about the line $y = x$, to reason "I have to have the geometric reflection of that point on my graph... and that forces it to be decreasing, because when I flip a point... well, if I flip a point above that *y* equals *x* line across that line, it moves to the right and down. And so there's a geometric argument that it has to be decreasing". After this geometric reasoning, he asserted, "I think I'm done." His inquiry then shifted into articulating an argument.

The linear algebra task asked participants to show that two similar 3x3 matrices would have the same characteristic and minimal polynomials. Dr. H set about the problem of determining why the statement should be true by sequentially proposing and assessing the following potential tools:1) if he had a theorem for determinants, 2) if he could argue that the polynomials are a property of the transformations in the change of bases, or 3) if he could "do something about row reduction preservation". He did not deem any of the tools as being immediately helpful. He transitioned into working a numeric example where he generated two similar 2x2 matrices and went about computing their characteristic polynomials to see why they would be the same. Due to a computation error, his polynomials appeared different. Dr. H chose not to inquire into where the error occurred because "even without looking for my error, I don't think that that's promising to do it from the definition." He then reflected back on the polynomials, recognized the roots would be eigenvalues, then decided he could construct a proof by reasoning about the eigenvalues. Dr. H then constructed an argument that required the eigenvalues to be distinct in order for his proof to work and claimed that he did not care enough to worry about the case they were not.

Once participants acquired conceptual insight into the problem, they switched to applying tools with the purpose of looking for technical handle, or a way to communicate the proof (Raman, et al., 2009; Raman & Weber, 2006). For example:

> Dr. N: Ok so, on the other hand if we start here and if we do something like that, can we make it be its own inverse? It just has to be symmetrical about that point. Ok, now at least I believe the statement... Ok so it must be monotonically decreasing, so now what could I do to give a proof of that? Well, I could try to... just kind of do a straightforward thing, say let *c* be less than *d*, and I want to show that, see decreasing... that *f* of *c* is greater than *f* of *d*.

For two of the participants, Dr. H and Dr. K, converting a pictorial or numeric argument to an analytic argument was not deemed necessary. Dr. H claimed he would not have continued with a proof if we had not asked for such; Dr. K claimed he was finished and did not need to write any more for a proof.

Discussion

In the two episodes where the participants searched for an insight into why the statement is true, we observed examples of the periods of "reflection, action, and evaluation" that characterize Dewey's theory of inquiry (Dewey, 1938; Garrison, 2009). The participants reflected on the problem, reflected on tools that they may choose, applied a chosen tool, and reflected on the effectiveness of the tools and how their implementation changed the problem. We hypothesize participants' lack of interest in producing an analytic argument may be a consequence of the situation of an interview or the mathematicians' attitudes of where the actual mathematics happens in constructing proof.

Questions for Discussion:

(1) What distinctions in tool use might be important in differentiating between looking for any conceptual insight and looking for an insight into the reasons why the statement is true that can lead to communicable argument?

(2) How might we triangulate our hypotheses about mathematicians' goals during proof construction in further studies?

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