EMERGENT MODELING AND RIEMANN SUM

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This research focuses on mental challenges that students face and how they resolve these challenges while transitioning from intuitive reasoning to constructing a more formal mathematical structure of Riemann sum while modeling "real life" contexts. A pair of Calculus I students who had just received instruction on definite integral defined using Riemann sums and illustrated as area under the curve participated in multiple interview sessions. They were given contextual problems related to Riemann sums but were not informed of this relationship. Our intent was to observe students' transitioning from model of to model for reasoning while modeling these problem situations. Results indicate that students conceived of five major conceptions during their first task and their reasoning from the first task that became a model for reasoning about their next task. In this paper we detail those conceptions and their reasoning that became model for reasoning on the second task.

Keywords: Emergent modeling, Riemann sum, Quantitative reasoning, Definite integral

Introduction and Research Questions

Riemann sums provide a foundation upon which one can understand why definite integrals model various situations. Previous research has detailed mental challenges that students face while reasoning about accumulation contexts, and has stressed how students could perform routine procedures for definite integral without being able to explain their reasoning (e.g., Artigue, 1991, Hall, 2010, Orton, 1983; Sealey, 2006). Research that has detailed how students might shift from more intuitive understanding to a more formal understanding has focused on roles of quantitative reasoning (Sealey, 2006; Thompson, 1994) and how that reasoning can support a more conceptually accessible formation of the Fundamental Theorem of Calculus (Thompson & Silverman, 2008). Other research has detailed the importance of conceiving of appropriate structural elements of the Riemann sum within context in order to complete approximation tasks (Sealey & Oehrtman, 2008). But when students come to understand Riemann sums as a model of a particular situation, how does their reasoning about that model influence their reasoning in constructing Riemann sum models of subsequent situations? This research attempts to answer the following questions. (1) What challenges do students face and how do they resolve those challenges as they constitute Riemann sum as a model of a contextual approximation problem? (2) How do students utilize their prior reasoning from their constitution of their Riemann sum model as a model for their reasoning about subsequent problems?

Theoretical Perspective and Methods

Rooted on the theory of Realistic Mathematics Education (Freudenthal, 1973), emergent modeling is an instructional design heuristic where modeling is viewed as an active organizing process where models co-evolve as students reorganize their intuitive reasoning and construct more formal mathematical reasoning (Gravemeijer 2002; Heuvel-Panhuizen, 2003). Models are viewed as more than representations but as holistic organizing activities including a solution strategy. *Model of* is the starting phase of emergent modeling where learners consider a model to be context-specific and employ informal solution strategies. Model for is the latter phase of emergent modeling where learners shift from thinking about the problem situation of the model to reasoning about it mathematically. "The model changes character, it becomes an entity of its own, and as such it can function as a *model for* more formal mathematical reasoning" (emphasis in original, Gravemeijer, 2002, p.2). Quantitative reasoning provides a means of modeling where students conceive of quantities, construct relationships between quantities, and meaningfully operate on those quantities that can support the construction of further quantities as one reasons with and about the problem situation (Larson, 2010; Thompson, 2011). When a conceived quantity is specifically attached to an attribute of a problem situation, any representing of this quantity would indicate model of reasoning, but as one reasons about this quantity within a quantitative structure without referring to a problem situation, that reasoning emerges as a model for their reasoning about the mathematics.

Ten interview sessions (50-148 minutes) were conducted with two volunteer Calculus I students, Sam and Chris (pseudonyms), who had been introduced to the definite integral through Riemann sums illustrated as area (Stewart, 2008). Students were given three approximation tasks related to Riemann sums, out of which two emphasized finding under and overestimates to total distance traveled based off of a table containing velocities and a velocity function, respectively (Figure 1). The third task was related to pressure on a dam, but this paper will focus only on the first two tasks since analysis of the third task is ongoing. For these two tasks, additional subtasks included drawing pictures of the actual situation, finding and illustrating error bounds, and graphing. Sessions were videotaped to analyze how students modeled their problem situations. Models were identified based on students' reasoning as exemplified by their representations and verbal utterances. When students directly related their reasoning to the problem situation, this was viewed as model of reasoning. Prior patterns of reasoning and representing when applied to a current problem situation were viewed as indicators of potential model for reasoning.

Task 1: The table below shows the velocity of a car travelling from Conway to Little Rock. In this activity you will approximate the distance travelled by the car during the first 10 seconds of the car entering the southbound I-40 ramp.

T(s)	0	2	4	6	8	10
V(ft/s)	0	21	34	44	51	56

Task 2: NASA's Q36 Robotic Lunar Rover can travel up to 3 hours on a single charge and has a range of 1.6 miles. After t hours of traveling, its speed in miles per hour is given by the function $v(t) = sin\sqrt{9 - t^2}$. In this activity you will approximate the distance travelled by the Lunar Rover in the first two hours.

Figure 1. First two teaching experiment tasks.

Results

The results reported here will focus on student's emerging model of Task 1 (Table 1) and reasoning about Task 1 that reappeared in Task 2 to suggest model for reasoning.

Initially, Sam and Chris realized that the varying velocities and the finite amount of data caused problems with easily completing Task 1. Reasoning from the provided table, their first conception of a distance/rate/time relationship (DRT 1[1]) was modeled as a picture containing snapshots of a car equally distanced between every two seconds (Figure 1, Picture a). After the

[1] Descriptions for DRT 1, DRT 2, DRT 3, Total 1, and Total 2 can be found in Table 1.

Table 1.
Distinct Conceptions During Task 1.

Conception	Description of reasoning
DRT 1: Distance changes as time changes	Omitted explicit detail to amounts of change in velocity. Pictorially represented as a vehicle with constant amounts of changes in distance per two second intervals.
DRT 2: Distance is change in velocity × change in time	Initially supported by their reasoning about amounts of change in distance vary because of changing velocities. Pictorially represented as a vehicle with decreasing amounts of changes in distance per two second intervals which became a model for distance as $d = \Delta V \cdot \Delta t$.
DRT 3: Distance is constant velocity × change in time	Initially only conceived for a vehicle traveling at constant velocities. After adjusting their picture to model a vehicle with increasing amounts of changes in distance and after "supposing" their vehicle as traveling at constant velocities was this conception applied to their context. Formulaically represented as $d = V_p \Delta t$.
Total 1: Total distance approximated by adding up distances are underestimates or inconclusive.	Adding up amounts of change in distances approximates total distance. Coordinated with DRT 2 and then DRT 3. With DRT 3 it was initially represented as $\sum_{p=0}^{5} V_P \Delta t$. For Sam, this sum was an underestimate because the sum would increase towards the exact total distance traveled as more data points were added. For Chris, this sum was inconclusive because the data table did not reveal what happened between data points.
Total 2: Total distance approximated by adding up using max. and min. velocities.	Coordinated with DRT 3. They conceived of maximum and minimum velocities over a time-interval as approximations to varying velocity over that interval. Underestimates and overestimates were represented by $\sum_{p=0}^{4} V_P \Delta t$ and $\sum_{p=1}^{5} V_P \Delta t$, respectively.

Note. DRT = Distance, Rate, and Time relationship.

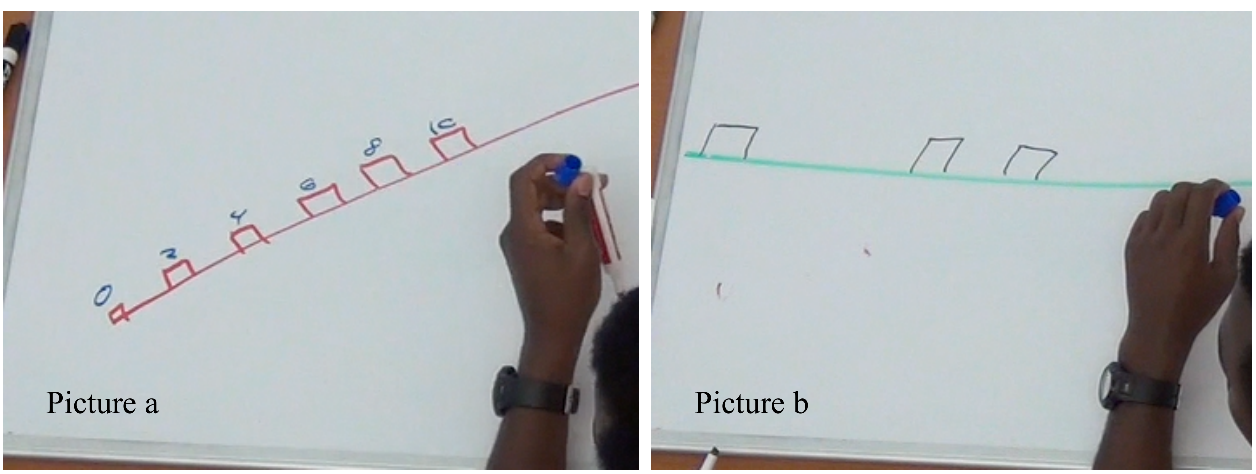

Figure 2. Two pictures showing locations of the car every two seconds.

facilitator prompted them to be "picky" with their picture, they attended to varying amounts of change in distance between snapshots, and represented this conception pictorially with increasing changes in distance between every 2-second snapshot (Figure 1, Picture b) and formulaically as "$d = \Delta V \cdot \Delta t$" (DRT 2). After prompted to think about a "real life" situation of a car merging onto an interstate, they adjusted their picture to indicate increasing distances between snapshots. By this moment they had indicated that adding up individual distances would provide approximations for total distance (Total 1). At first, DRT 3 appeared in response to an additional facilitator question concerning another situation in which a car traveled at 70 mph for two hours and 80 mph for one more hour. Though they concluded that the car in the other situation traveled 220 miles, DRT 2 persisted in their reasoning about the car with varying velocities in Task 1. Once they realized that their formula $d = \Delta V \cdot \Delta t$ for the other situation yielded a conflicting answer when applied to the additional question did they rethink DRT 2. Attempting to calculate error bound, they grappled with finding both under and overestimates for total distance. After three hours since starting this task, once they had conceived of the roles of maximum and minimum velocities as approximations for varying velocities over 2-second intervals, they coordinated DRT 3 with total distance and were able to find both under and overestimates for total distance (Total 2). Later Sam compared getting an exact distance to a perfect video, “We have an infinite number of snapshots, […] a solid image of what- We have a video, a perfect video where there is no frames or anything like that, an ideal video.” They finished with representations seen in Figure 3.

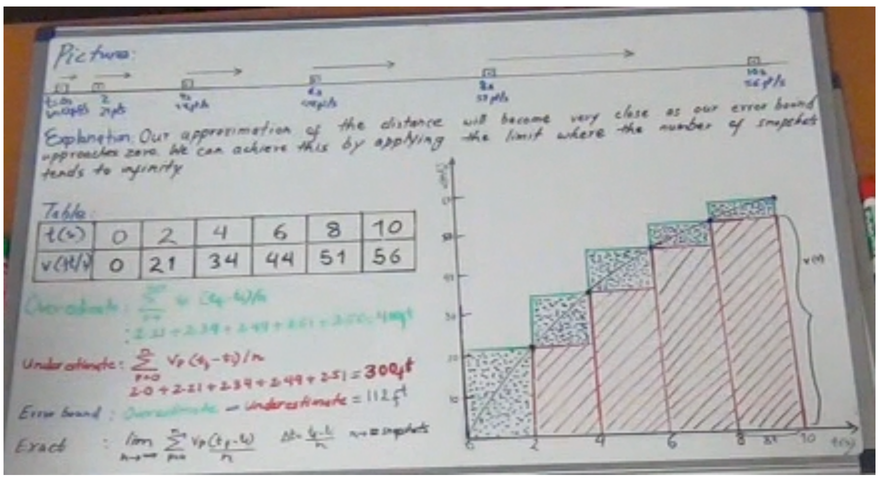

Figure 3. Chris and Sam's multiple representations of Task 1

Immediately after being given Task 2, Sam asked Chris, “Don’t you think the picture looks the same like last time?” He then drew a picture with snapshots of the rover at every half hour interval. To understand the changing velocity of the rover, they constructed a table of values similar to the given table in Task 1. Once their table was constructed, they explicitly quantified amounts of change and coordinated that with labels added to their picture. After being asked, “Where is distance?” they noted distances between snapshots on their picture and proclaimed that finding total distance was, “the same as what we did last time.” Subsequently, they represented total distance as $\sum V_{(t)} \Delta t$. Although imprecise, this representation captures the multiplicative structure between particular velocities and amounts of change in time within a summation. In the process of constructing a numerical approximation to total distance travelled, they employed their prior reasoning concerning maximum and minimum velocity over an interval and coordinated their DRT 3 and Total 2 to calculate under and overestimates, eventually represented as summations with appropriate adjustments to the starting values of the index. From approximations to exact distance, Sam stated, “We go from having pictures, to a flip book, to a video, to like one true continuous string where there is no frame rate.” Chris generalizes, “all of these summarize how you can make an error smaller by increasing the number of snapshots...as we increase the number of snapshots we tending to get the exact displacement so, that's what both of them summarize [pointing to both Task 1 and 2]."

Discussion and Questions

We observe that the challenges presented by Task 1 were not easily overcome by Sam and Chris but their engagement of these challenges supported them in forming patterns of reasoning for more effectively modeling Task 2. For instance, Sam and Chris had to construct appropriate ways for reasoning about a relationship between distance, rate, and time for a car of increasing velocity. They had to conceive of pertinent roles for minimum and maximum velocities for under and overestimates, and relate those to a notion of summing up distances to obtain Riemann sums for under and overestimates. For instance, Sam and Chris' conceptions of minimum and maximum velocities within a model of calculating under and overestimates for total distance during Task 1 was first represented after three hours of work. In contrast, they readily represented these estimates for total distance for Task 2 within thirty-two minutes. How were they able to progress so rapidly during Task 2? Their picture, graph, table, and formulaic expressions from Task 1 served as reference points for them to make connections between their two tasks as they conceived of, represented, and related relevant quantities. For example, before they firmly committed to using their reasoning from Task 1 applied to Task 2, their pictures and tables supported their conceiving of varying velocities, amounts of change in time, amounts of change in distance, and in relating these quantities while building connections across the tasks. As these connections became more apparent, the students progressed in constructing appropriate Riemann sum approximations. We note that it was not merely the end results of Task 1, but elements of their reasoning that went behind creating those end results, including a solution strategy, which served as a model for their subsequent reasoning during Task 2.

We acknowledge that our work with one pair of students does not necessarily generalize to others. Furthermore, the model for reasoning being reported may be more general reasoning that is still tied to Riemann sum approximation problems involving relationships between distance, rate, and time. We also note that since the students were exposed to Riemann sums, they were not reinventing Riemann sum symbolizations but were conceiving of a multiplicative structure within contexts and constructing relationships between this structure and some existing Riemann sum structure. Our questions are: How can we design tasks to better capture students' modeling activities and their transition from model of to model for in the context of definite integral and Riemann sum within a research context? For students who have not been exposed to Riemann sums, how can we modify our tasks to generate an intellectual need for these sums and subsequently support these students in constructing a Riemann sum? How might activities be effectively scaffolded to support the model of / model for transition in a classroom?

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