# IMPLICATIONS OF REALISTIC MATHEMATICS EDUCATION FOR ANALYZING STUDENT LEARNING

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*The primary goal of this work is to articulate a theoretical foundation based on Realistic Mathematics Education (RME) that can support the analysis of student learning, both individual and collective, by documenting changes in local activity. To do so, I will build on previous work on the analytic implications of the Emergent Perspective, specifically Rasmussen and Stephan's (2008) analytic approach to documenting the establishment of classroom mathematical practices. The Emergent Perspective is broadly consistent with RME, but the existing analytic methods related to the Emergent Perspective fail to draw on the theoretical constructs provided by RME. For instance, current analytic methods fail to draw on the RME Emergent Models heuristic to inform the analysis of the development of mathematical practices related to models of/for student mathematical activity. Here I will be explicitly considering the roles that RME constructs could play in analytic processes consistent with the Emergent Perspective.*

Key Words: Realistic Mathematics Education, Learning, Mathematizing, Emergent Models, Analytic methods

As I worked to investigate the implementation of an inquiry-oriented abstract algebra curriculum (Johnson & Larsen, 2012; Johnson, 2012) I became increasingly aware of the need for a comprehensive approach to making sense of student learning in this context. In particular, since the curriculum was designed based on Realistic Mathematics Education (RME), it was apparent that analyses of student learning should draw on the theoretical constructs that comprise RME. However, the existing formal analytic approaches do not do so, and the current formulations of RME are not articulated in a way to support the development of such approaches.

In order to articulate RME in such a way that supports new analytic techniques, I will coordinate the various RME constructs involving levels (of generality) of activity. For example, the Emergent Models design heuristic features a transition from what is called referential activity to general activity (Gravemeijer, 1999). I will look at the role of vertical and horizontal mathematizing (Gravemeijer & Doorman, 1999) in such a transition and consider how chains of signification (Gravemeijer, 1999) can facilitate this transition on the local scale.

Based on the Emergent Perspective (Cobb, 2000), Rasmussen and Stephan (2008) have developed an analytic scheme to document the development of classroom mathematical practices. Here I will consider how such an analytic scheme may be refined or expanded by drawing explicitly on various RME design heuristics related to shifts in levels of activity.

**An RME Characterization of Student Learning**

From an RME perspective it makes sense to conceive of learning as the creation of new mathematical realities. RME is an instructional design theory that is grounded in the belief that formal mathematics can be developed by engaging in mathematical actives, where these activities serve to progressively expand students' common sense. As described by Gravemeijer (1999), "what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality" (p. 156).

The students' experiential reality includes what the students can access on a "commonsensical level".

> "Real" is not intended here to be understood ontologically (whatever ontology may mean), therefore neither metaphysically (Plato) nor physically (Aristotle); not even, I would even say, psychologically, but instead commonsensically as … meant by the one who uses the term unreflectingly. It is not bound to the space-time world. It includes mental objects and mental activities. (Fredunethal, 1991, p. 17)

Therefore, the problem context for RME based curriculum need not be "real" in the sense that the students would access such scenarios in their everyday life. Instead, the students only need to be able to access the problem context on an intuitive level. In this way a magic carpet can be understood as experientially real, even though it is not physically real. As such, a context based on the movements of a magic carpet may form the foundation for the reinvention of formal mathematics (Wawro et al., 2012).

Within an experientially real context, RME based curriculum presents instructional tasks that promote mathematizing the problem context. This activity of mathematizing, "which stands for organizing from a mathematical perspective" (Gravemeijer & Doorman, 1999, p. 116), is a central mathematical activity in RME based curriculum and can be used to explain the learning process.

> In this view, students should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics. (Gravemeijer, 1999, p. 158)

It is through this cycle of progressive mathematizing, between mathematizing reality and mathematizing mathematical activity, that students reinvent mathematics by expanding their mathematical reality.

Initially, as students mathematize their experiential reality, they are engaging in horizontal mathematizing. Horizontal mathematizing includes activities such as translating, describing, and organizing aspect of problem context into mathematical terms (Gravemeijer & Doorman, 1999, p. 116-117). For instance, students may be asked to describe and devise a set of symbols for the symmetries of an equilateral triangle (Larsen, 2012). In this way, "horizontal mathematisation leads from the world of *life* to the world of *symbols*" (Fredunethal, 1991, p. 41). The artifacts of horizontal mathematizing include inscriptions, symbols, and procedures.

While horizontal mathematizing is a crucial step in the reinvention process, reinvention "demands that the students mathematize their own mathematical activity as well" (Gravemeijer & Doorman, 1999, p. 116-177). Vertical mathematizing characterizes activities through which students mathematize their own mathematical activity and may include generalizing, defining, and algorithmatizing (Rasmussen et al., 2005). For instance, for an RME inspired differential equations course, Rasmussen et al. (2005) describe a scenario in which students are first asked to approximate the number of fish in a pond given various initial input values. This task resulted in the students generating inscriptions during their initial horizontal mathematizing (i.e. tables and graphs that recorded their results). The students were then asked to reflect on their previous work in order to generate an algorithm by which approximations may be found regardless of the initial constraints. This task necessitates that the students engage in vertical mathematizing, as the students needed to mathematize their previous mathematical activity. Further, the algorithm that was generated through this vertical mathematization is now available to the students for further mathematization. In that sense, an artifact of vertical mathematization can become part of the students' expanding mathematical reality.

Therefore, student learning can be understood as the incorporation of new mathematics into the student's expanding mathematical reality, where this reality expands through the process of progressive mathematization. As Gravemeijer and Doorman (1999) state, "it is in the process of progressive mathematization - which comprises both the horizontal and vertical component - that the students construct (new) mathematics" (p. 116 – 117).

Notice that, as students' activity shifts from horizontal to vertical mathematizing, there is a shift in the generality of the student activity. Initially, horizontal mathematizing is limited to the specific problem context. As students transition to vertical mathematizing, this specific problem context is no longer the focus of the activity, rather the students mathematize their own mathematical activity to support their reasoning in a different or more general situation. In this way, vertical mathematizing may involve activities such as abstracting, generalizing, and formalizing (Rasmussen et al., 2005). One way that the RME guides the design of instruction intended to support these shifts in generality is through the emergent models heuristic.

**Emergent Models**

The RME Emergent Models instructional design heuristic is meant to promote the evolution of formal mathematics from students' informal understandings through the development of models (Gravemeijer, 1999). Models are defined as "student-generated ways of organizing their activity with observable and mental tools" (Zandieh & Rasmussen, 2010, p. 58).

Concepts that first emerge as a *models-of* student activity become *models-for* more formal activity. Gravemeijer (1999) describes four layers of activity. Initially student activity is restricted to the *task setting*, where their work is dependent on their understanding of the problem setting. *Referential* activity develops as students construct models that refer to their work in the task setting. *General activity* is reached when these models are no longer tied to the task setting. Finally, *formal activity* no longer relies on models. In regards to these four levels of activity, the shift from *model-of* to *model-for* occurs as students shift from *referential activity* to *general activity*.

It is at this shift between referential and general activity that the model transitions from the result of mathematizing the problem context into an object which itself can be the basis for further mathematizing. On the global scale, transitioning between any two levels of activity can be interpreted as the result of vertical mathematizing. (Whereas activity within a single level of activity can be understood as horizontal mathematizing.) In particular, the activity that supports the transition between a *model-of* to a *model-for* is a particularly significant example of vertical mathematizing as the activity shifts from referential to general.

Gravemeijer (1999) concedes that while the "model" is a global overarching concept, in practice the "model manifests itself in various symbolic representations" (p. 170). The construct of a *chain of signification* provides one way to describe changes in the symbolic representation of the model during the instructional sequence. Central to the chain of signification construct is the idea of a sign, which is made up of a signifier (a name or symbol) and the signified (that which the signifier is referencing, such as the students' activity). As the chain builds, previous signs can become the signified in subsequent signs. In this way, student activity can become an object that a signifier references. In this way, a chain of signification accounts for reification on a local scale. These local changes then support the reification of the global model.

While a chain of signification looks at reification on a local scale, record-of/tool-for serves as a way to understand how result of an activity (a record-of) is used in further mathematics (tool-for). As described by Larsen (2004), an inscription representing students' mathematical activity transitions from a *record-of* to a *tool-for* when the students use the notational record to achieve subsequent mathematical goals. Therefore, instead of focusing on the relationships between the students emerging symbols and notations (as with chains of signification), the record-of / tool-for construct focuses on changes in how the emerging symbols and notations are used.

Gravemeijer (1999) notes that, " the shift from model of to model for is reflexively related to the creation of a new mathematical reality" (p. 175). On the one hand, the transition from a model-of to a model-for reflects a transition of the model as a product of student activity to an instrument for supporting more formal mathematical reasoning. Therefore, the of/for transition serves to expand the mathematical reality. On the other hand, the creation of a new mathematical reality (as understood as local shifts from record-of activity to tool-for further mathematizing) aids in the transition of the global model. As a result, the emergent model construct offers a promising starting point when trying to document the development of a new mathematical reality.

**Implications for Analyzing Student Learning**

If an instructional sequence were designed to promote the reinvention of a concept by way of an emergent models transition, then one would want an analysis of students' learning to draw on the emergent models construct. So if the goal is to promote a shift from model-of to model-for, then an analysis of students' activity should explicitly draw on theoretical constructs related to such shifts. Here I will consider Rasmussen and Stephan's (2008) analytic framework for documenting the development of classroom mathematical practices (Cobb, 2000) before presenting a revised methodology that explicitly draws on the emergent model construct.

The Emergent Perspective is a framework for analyzing individual and collective mathematical activity in classroom settings. Cobb (2000) describes the Emergent Perspective as an "interactionist perspective on communal classroom processes and a psychological constructivist perspective on individual students' activity as they participate in and contribute to the development of these collective processes "(p. 321). Generally speaking the Emergent Perspective and RME are consistent, as "both content that mathematics is a creative human activity and that mathematical learning occurs as students develop effective ways to solve problems and cope with situations. Further, both propose that mathematical development involves the bringing forth of a mathematical reality" (p. 317). One analytic methodology situated within the Emergent Perspective, and used to analyze student learning in RME based contexts, is Rasmussen and Stephan's (2008) process for documenting of the development of classroom mathematical practices.

Rasmussen and Stephan (2008) define a classroom mathematical practice as "a collection of as-if-shared ideas that are integral to the development of a more general mathematical activity" (p. 201). The example presented by the authors was the classroom mathematical practice of "creating and organizing collections of solution functions" (p. 201), which entails four related taken-as-if-shared ways of reasoning about graphs and functions. Rasmussen and Stephan's (2008) methodology for documenting the development of classroom mathematical practices can be understood as a process for identifying changes over extended classroom sessions, where these long term changes are identified by looking for local shifts in the classroom's normative ways of reasoning.

To document the development of such a classroom mathematical practices (and by necessity the normative ways of reasoning that comprise them), Rasmussen and Stephan outlined a three-phase approach based on the idea that "learning is created in argumentation" (p. 197). Accordingly, Rasmussen and Stephan's methodology documents the evolution of collective argumentation by tracking the claims, data, warrants, and backings provided during classroom discussions.

Rasmussen and Stephan (2008) identified two local shifts in classroom argumentation that signify that a mathematical idea has become a normative way of reasoning. First, if arguments no longer require warrants or backing by the community, then the mathematical idea is considered to be taken-as-if-shared. Second, if any part of an argument (data, claim, warrant, backing) is used in a different way in a new argument and is unchallenged, then the mathematical idea that shifted positions is considered to be taken-as-if-shared.

I propose that, when documenting the development of classroom mathematical practices related to *models of/for* student mathematical activity, one can similarly look for local changes in the students emerging symbols and notations. These local changes can either be 1) in the relationships between the students emerging symbols and notations, as described by the chains of signification construct, or 2) in how the emerging symbols and notations are used, as described by the record-of/tool-for construct. Here I will briefly present an example of each by drawing on an inquiry-oriented abstract algebra curriculum (Larsen, Johnson, & Bartlo, 2012) and discuss how each can be seen an analogous to aspects of Rasmussen and Stephan's (2008) methodology.

The abstract algebra curriculum launches in the context of symmetries of an equilateral triangle. Initially, the students begin by physically moving a triangle in order to identify the six symmetries of an equilateral triangle. The students are then asked to represent these six symmetries with a diagram, a written description, and a symbol (see figure 1). This set of inscriptions can be thought of as a signifier that signifies the students' activity of manipulating the triangle. The students are then asked to generate a new set of symbols, this time representing each symmetry in terms of a vertical flip, *F*, and a $120^{o}$ clockwise rotation, *R*. This new set of symbols represents the next step in the chain of signification, with the earlier sign "sliding under" this subsequent sign. The original sign, which was comprised of both the students' initial signifier (i.e. their initial inscriptions) and the original signified activity (i.e. physically manipulating the triangle), is now signified by this new set of symbols. So as the chain of signification builds, students no longer need to directly consider the original activity of manipulating the triangle. For example, when working with symbols expressed in terms of *F* and *R*, students may no longer need to keep in mind that they refer to motions of a triangle. In this way, one sign "sliding under" to become signified in a subsequent sign can be seen as analogous to the dropping of warrants and backings in Rasmussen and Stephan's (2008) argumentation methodology – as in both cases an idea/inscription that used to be seen as necessary drops away (but can be retrieved if needed).

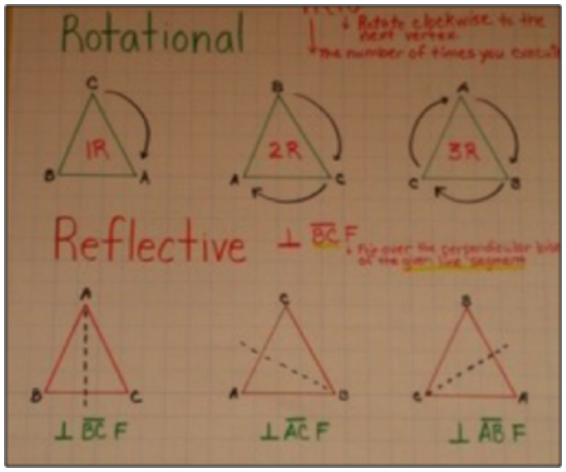


Figure 1. Pictures and Initial Symbols for the Symmetries of an Equilateral Triangle

Once the classroom develops a common set of symbols using *F* and *R*, the students are asked to consider any combination of two symmetries. An operation table initially emerges as a *record-of* the students' activity. Later, as the students argue that the identity element of a group must be unique, some students draw on the operation table as a *tool-for* constructing a proof (for a full description see Larsen, 2009). So, an inscription that first served to record the students' mathematical activity later served as a tool for subsequent mathematical activity. I argue that this shift is analogous to that represented by a claim from one argument becoming a warrant in a later argument (Rasmussen and Staphan's (2008) second criteria). In each case, the role of the idea/inscription changes in an important way – specifically the role of the idea/inscription shifts from being a product (e.g., the claim of an argument or a record) to becoming an instrument (e.g., the warrant of an argument or a tool).

Thus, like Rasmussen and Stephan (2008), I propose documenting local shifts in order to accumulate evidence of global transitions. Further, the local shifts I propose for documenting the emergent models transition can be understood as analogous to the kinds of shifts in argumentation that Rasmussen and Stephan took as evidence for the development of classroom mathematical practices. Therefore, I see this work as a generalization of the principles that underlie their method and hence as a coherent starting point for integrating the constructs of RME with the analytic framework provided by the Emergent Perspective.

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