

ASSESSMENT OF STUDENTS' UNDERSTANDING OF RELATED RATES PROBLEMS

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This study started with a thorough analysis of student work on problems involving related rates of change in a first-year differential calculus course at a large, research-focused university. In two sections of the course, students' written solutions to geometric related rates problems were coded and analyzed, and students' learning was tracked throughout the term. Three months after the end of term, "think-aloud" interviews were conducted with some of the students who completed the course. The interviews and some of the written assessments were structured based on the classification of key steps in solving related rates proposed by Martin (2000). Our preliminary findings revealed a widespread, persistent use of algorithmic procedures to generate a solution, observed in both the treatment of the physical and geometric problem, and the approach to the differentiation, and raised the question of whether traditional exam questions are a true measure of students' understanding of related rates.

Key words: Related rates, Calculus assessment, Misconceptions

Introduction and Research Questions

In many traditional differential calculus courses in North American universities, after learning about rates of change and various techniques of differentiation, students learn to apply these ideas to solve related rates problems, that is, problems that require the evaluation of "the rate of change (with respect to time) of some variables based on its relationship [often geometric in nature] to other variables whose rates of change are known" (Dick & Patton, 1992). Existing research on students' difficulties with these problems indicates that students lack conceptual understanding of variable and have trouble in distinguishing between variables and constants (White & Mitchelmore, 1996, Martin, 2000), as well as trouble in engaging in covariational reasoning (Engelke, 2004). A classification of the main steps in solving geometric related rates problems was proposed by Martin (2000), who discusses the results of assessing students on the specific steps, reporting greater correlations between procedural knowledge and success at solving related rates problems, while Engelke (2007) discussed a possible framework to describe how a mental model for a related rates problem is developed during the solution process.

Being a classic course topic, related rates problems were chosen as the setting for a classroom experiment that took place in 2011 in two sections of a large calculus course (Code et al. 2012). As part of that project, test items similar to traditional exam questions were developed to assess students' skills at solving related rates problems. The detailed analysis of student work performed in that study brought to light specific limitations of these assessment tools and questioned the effectiveness of traditional exam questions as an accurate measure of understanding of related rates. Motivated by these findings, we conducted a follow-up study aimed at deepening our understanding of students' difficulties with related rates. Using a similar framework to that presented by Martin (2000), we assessed students' mastery of specific steps in solving related rates problems, extending her methodology with the use of student interviews. The main goal of this study is to investigate the following questions:

What are the sources of common misconceptions observed in students' solutions to related rates problems on written exams?

Do traditional exam questions involving related rates accurately assess students' understanding of such topic?

Methodology

Written solutions of geometric related rates problems from four different assessments were collected for $N = 300$ students enrolled in a large Calculus 1 course at a research-focused university. The course is primarily aimed at Business and Economics majors with some prior knowledge of calculus (high school calculus), but it shares most core material with the science-oriented Calculus 1 offered at the same institution (about a third of its student population are in fact science majors). Our sample represents about 25% of the total course enrolment, and was selected from two of the 11 course sections. Student work was collected at four different stages during the term: on a short diagnostic test at the beginning of the term, a quiz at the end of the week of instruction on related rates problems, a midterm exam two weeks later, and a final exam at the end of the course. Both the midterm and the final exams accounted for a portion of the final grade, while the diagnostic and the quiz were part of a number of in-class activities that were worth a small fraction of the final grade (1%), awarded based on participation. About three months after the end of the course, "think aloud" interviews were conducted with 11 students randomly selected from the original sample.

Preliminary Results

From the analysis of students' written work and the tracking of performance over the term, we observed significant improvements of key skills in solving related rates as a result of both instruction and feedback from tests. After targeted instruction and homework involving related rates problems, the majority of students showed improved ability in performing the early steps of a solution compared to their incoming skills at the beginning of term. Differentiation, however, appeared to be one of the major stumbling blocks for students. Despite several weeks of review and practice of the basic concepts and rules of differentiation, when students start to work with related rates they had not yet developed the sufficient skills to carry out sophisticated calculations such as the derivative (with respect to time) of a functional expression containing more than one time-dependent variable, like for example the function representing the volume of a growing cone. Skills improved over the course of the term, but these difficulties were not fully resolved by the end of the course, and in some cases persisted beyond the end of the course, as confirmed by the student interviews. A preliminary analysis of student thinking observed in the interviews would suggest that the source of these difficulties stems from lack of a deep understanding of the differentiation process, rather than some misunderstanding of the specific physical problem at hand. Interestingly, to bypass the challenge posed by these complicated functional expressions, instructors and textbooks often teach students to reduce the number of variables by performing an appropriate substitution before taking the derivative. While this strategy simplifies the problem significantly for students, the data we collected suggest that proficiency in implementing this solution strategy is likely an indication of procedural knowledge rather than conceptual understanding, raising the question of whether testing the students on how proficient they are in providing written solutions for these problems is a true measure of their understanding of related rates.

Discussion Questions

Do students really possess the technical skills to handle the mathematical sophistication that related rates problems present?

Are traditional questions testing the ability to generate a full, correct solution a true measure of students' understanding of related rates?

What assessment strategies can be developed to effectively measure understanding of related rates?

References

- Code, W., Kohler, D., Piccolo, C., MacLean, M. (2012). Teaching Methods Comparison in a Large Introductory Calculus Class. *RUME XV Conference Proceedings*.
- Dick, T. P. & Patton, C. M. (1992). *Calculus (Vol 1)* Boston.
- Engelke, N. (2004). Related rates problems: Identifying Conceptual Barriers. *Proc. of 26th Conference of the North-American chapter of the International Group of the Psychology of Mathematics Education* (pp. 455-462). Toronto, ON: OISE.
- Engelke, N. (2007). A Framework to Describe the Solution Process for Related Rates Problems in Calculus. *RUME X Conference Proceedings*.
- Martin, T. (2000). Calculus students' ability to solve geometric related-rates problems. *Mathematics Education Research Journal*, 12(2), 74-91. doi:10.1007/BF03217077
- White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education*, 27(1), 79-95.