CRITIQUING THE REASONING OF OTHERS: DEVIL'S ADVOCATE AND PEER INTERPRETATIONS AS INSTRUCTIONAL INTERVENTIONS

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This study investigated the ways in which college mathematics teachers might encourage the development of student reasoning through critiquing activities. In particular, we focused on identifying situations in which the instructional interventions were implemented to encourage the critiquing of arguments and in which students explained another's reasoning. Data for the study come from two teaching experiments – one from the domain of combinatorics and the other from real analysis. Through open coding of the data, Devil's Advocate and Peer Interpretations emerged as effective interventions for the creation of sources of perturbation for the students and for assisting in the resolution of a state of disequilibrium. These two interventions differ in design and in the type of reasoning students evaluate, but they both provoke students to further develop their reasoning, and therefore their understanding. We discuss the implications of these interventions for both research and teaching practice.

Key words: Combinatorics, Perturbation, Real Analysis, Student Reasoning, Teaching Practice

Introduction and Research Questions

The purpose of this study is to examine how mathematics teachers may leverage college students' reasoning and understanding of advanced mathematics by engaging them in critiquing the reasoning of others. Critiquing activities may perturb students who are not familiar with the ideas employed by the argument given or those whose reasoning might have a different base. It is hoped that through such activities students can grasp the essence of the argument, develop their ability to distinguish correct logic from flawed, and explain the flaws if they exist. Hence activities of critiquing others' arguments would provide students a way to resolve their perturbation and to develop their mathematical reasoning. In fact, it might be difficult for a student to find flaws in his or her own reasoning. However, critiquing arguments written by others or fictional characters might enable a student to reflect on his or her own reasoning and understanding. In line with the standpoint, the recently released Common Core State Standards (CCSS, 2011) state that mathematically proficient students of all levels develop the skills to construct viable arguments and critique the reasoning of others. The mathematics education community also has demonstrated increasing interests in using critiquing activities as a research method (Lockwood, 2011; Selden & Selden, 2003) or an instructional intervention (Halani, 2012; Kasman, 2006; Roh & Lee, 2011).

This paper focuses on teachers' creation of the source for student perturbation and their ways to facilitate students' understanding of new ideas through critiquing activities in the domains of combinatorics and real analysis. We consider that such understanding is a result of what Steffe et al. (1983) called "second-order models." This refers to an observer's model of the subject's knowledge, or the models that "observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (p. xvi). While a body of research explored teachers' models of their students' mathematics (e.g. Courtney, 2010; Silverman & Thompson, 2008), this study emphasizes students' construction of the second-order models of their peers or fictional characters and teachers' role in the development of such a construction. This study addresses the following research question: How might an instructor use critiquing activities to create sources of potential student perturbations along with the ways to resolve such perturbations?

Theoretical Framework

Under the perspective of constructivism (Von Glasersfeld, 1995) adopted in this study, the role of an instructor is to orient students' cognitive processes and aid students with their construction of mathematics. One way that a teacher might do exactly this is to create sources of potential perturbation along with ways to resolve it in order to encourage students to develop their reasoning, which is what we call *instructional provocations* in this paper.

Devil's Advocate (DA) and Peer Interpretations (PI) are two instructional provocations which extend Rasmussen and Marrongelle's (2006) "generative alternatives" and which are designed to encourage student explanation and justification. The first, *Devil's Advocate* (DA), refers to an incorrect or atypical argument provided to students by the instructor for evaluation. The purpose of this provocation is to highlight cognitive conflicts or to raise awareness of certain aspects of a topic. The students would refute the argument if they disagree or provide justification for parts of the argument otherwise. The second, *Peer Interpretations* (PI), refers to a student's interpretation of a peer's argument, at the prompting of the instructor. The purpose of this provocation is to highlight similarities and differences in thinking and to allow students to learn from each other. The student interpreting the peer's argument would often include his or her own reasoning and thinking into the interpretation.

Both DA and PI have the potential to create sources of perturbation or help resolve such perturbation. However, DA and PI differ from two aspects: First, when these provocations are designed is different – while there is often a predefined argument with using DA, PI comes about in the classroom as the interactions between students dynamically develop. Second, the types of reasoning presented are usually different. Since DA is typically prepared by the instructor prior to the lesson, DA is well-formed so that students analyze the essence of an argument. On the other hand, through PI, students analyze their peer's reasoning which is often in a formative stage so that students pull out the essence of the argument.

Methods and Analysis

Data for this study come from two teaching experiments conducted at a large southwestern university - one from the domain of combinatorics, the other from real analysis. Each teaching experiment constituted of 9 or 10 teaching sessions, and involved a teaching agent and two high-performing undergraduate students, with no prior experience in the subject. The rationale for including two teaching experiments is straight-forward: The content of mathematics courses typically involves the study of either discrete or continuous structures, yet in both cases, it is imperative to push students to further develop their reasoning and understanding by critiquing the reasoning of others.

The analysis of the data was conducted in several phases. Content logs and full transcripts of all videos were first created. We, as the research team, then used an open coding system (Strauss & Corbin, 1998) to identify situations where a student explained someone else's reasoning. Following this, we used open coding to classify the instructional interventions (e.g., DA and PI) which had the intention of encouraging students to critique the reasoning of others. The coded data were then reviewed for consistency.

Results

We found that in both teaching experiments, each instructor pushed a student to further develop his or her reasoning and understanding by critiquing the arguments of others. In particular, two provocations, DA and PI, emerged from the data analysis. Both DA and PI were implemented throughout the study, but only a few illustrative examples are provided here.

Case 1: PI in the Combinatorics Study - The ARIZONA Activity

In the combinatorics teaching experiment, the instructor, the first author of this paper, asked Kate and Boris to determine the number of ways to rearrange the letters in ARIZONA. First, Kate explained her idea:

"*I disregarded the facts that there's a repeated letter and I just said 'how many ways can* […] *you arrange these seven letters?' and that's going to be 7!. But, um, you're going to have to take some of those out.* […] *I think for every* […] *one possible order of the letters, you're going to have another* […] *that's the same because there's only one letter that is repeated. So like, if we had like just a random RZIANOA there's going to be two ways. By this, there's 7!, which count that* [RZIANOA] *twice. So I think you just divide 7! by 2 to take those out.*"

Kate determined her solution of 7!/2 by first imagining that she was permuting 7 distinct letters, though she did not use the term "distinct." She recognized that the repeated A's would actually mean that she had counted twice as many permutations as she wanted. Boris also tried to permute distinct objects first, but he tried to take away one of the A's before permuting the other 6. He then tried to insert the remaining A into the permutations he had just created, determining a solution of $6!\times 6$. The instructor asked Boris to explain Kate's argument. He responded, "*Well she went and found the total number of ways that you could arrange seven unique letters, which would be seven factorial, and she said that for each of those* […] *you're counting twice as many possibilities as you should, because of the two different A's you're assuming that those are unique letters. Like A_1 or A_2 when they're really just both A's. So you have to take out half of those*."

Notice that Boris did not repeat Kate's reasoning verbatim and instead reinterpreted it while adding further justification, thus indicating that he had built a second-order model of his peer's argument in order to extract its essence. Boris experienced disequilibrium when he realized that the two solutions the students had created could not both be correct and indicated that he believed Kate's argument to be correct by stating that he was not sure what he was counting twice. Boris eventually resolved his perturbation by recognizing mistakes in his own argument through comparing it with Kate's idea for dealing with duplicates. Thus it seems as if the instructor's request that Boris explain Kate's argument was an effective implementation of PI – it not only created a source of perturbation, but helped Boris resolve it as well. In the next session, Boris assimilated this same way of thinking in order to determine that there were 4!/2 ways to permute two blue, one red, and one black counter.

Case 2: PI in the Real Analysis Study – Proofs involving Inequalities

The students in the real analysis study, Sam and Jon, attempted to prove "For any $a,b\in\mathbb{R}, |a-b|\geq||a|-|b||$." In order to do so, they were directed towards first proving two lemmas: "Let $a,b\in\mathbb{R}$, then (1) $|a|-|b|\leq|a-b|$ and (2) $-(|a|-|b|)\leq|a-b|$." The students were already familiar with the triangular inequality and what they called Theorem 1 (*iii*): "Let $a,b\in\mathbb{R}$, then $|ab|=|b|\cdot|a|$." After the students had written a proof for Lemma 1 together, they spent some time thinking about Lemma 2 separately. Sam's written work can be seen in Figure 1. He wrote down Lemmas 1 and 2 on top of his paper, along with a theorem he thought he might want to use T2 (*ii*) in Figure 1. This written scratch work hinges on the idea that $|b|=|b-a+a|\leq|b-a|+|a|$ by the triangular inequality ("TE" in Figure 1). Sam stated that $|b-a|=|a-b|$ by Theorem 1(*iii*) and by the fact that $b-a=(-1)(a-b)$. After Sam explained his thought process, the instructor of the session, the first author of this paper, asked Jon to explain what Sam had just said. Jon indicated that he thought Sam had used Lemma 1, but Sam interrupted and pointed out that he had applied the triangular inequality. When the instructor asked Jon to state his understanding of Sam's argument, Jon realized that he did not fully understand Sam's argument. In fact, in his reflection that evening, Jon wrote, "[*the instructor*] *did ask us to explain our thinking several times, to articulate our logic. This helped me see some holes in my understanding. For example, before she asked me to explain what Sam did for Lemma* [2], *I thought he was manipulating* *the inequality in order to use Lemma 1. Instead, he was using the "adding zero" technique and applying the triangular inequality in order to set up it up for Theorem* [1(*iii*)].” This exemplifies a case when the instructor requested Jon his interpretation of Sam’s argument, Jon realized that his model of Sam’s argument was not what Sam intended. The instructor’s implementation of the PI was therefore effective in creating Jon’s perturbation which was later resolved as the two students collaboratively worked to complete their proof of the lemmas.

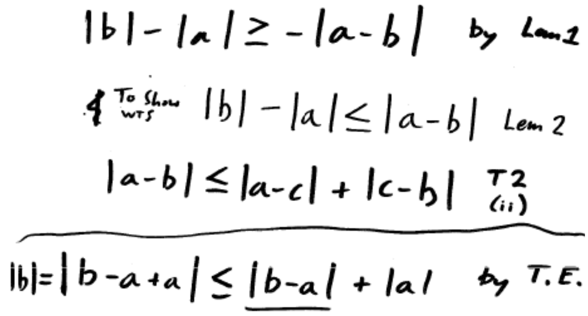

Figure 1: Sam's scratch work to prove Lemma 2: $-(|a|-|b|) \leq |a-b|$

Case 3: DA in the Real Analysis Study - The Vice of Inequality

At the third session of the real analysis study, the instructor, the third author of this paper, presented an alternative argument to the students in an attempt to highlight the importance of the order of quantifiers. First, she asked Sam and Jon the following question: “Would there be $x \in \mathbb{R}$ satisfying $\forall \varepsilon > 0, |x| < \varepsilon$?” After the students were given a few moments to think, the instructor asked Sam to share his thoughts. He responded “*Okay, so I was thinking that the only x that will work for this* […] *would be 0, because* […] *you could get x really small* (pinches fingers together)*, give it a really small value, but it's still not gonna work for any ε greater than 0 because the limit of that is 0. So, it's* [*x* is] *always going to be infinitessimally larger than 0, which means it* [ε] *can always be smaller than any positive x* [which is a contradiction]. *So, it would only work for 0.*” The instructor then presented an alternative solution to the given statement, asserting that there are infinitely many possible values of *x* based on the following theorem: If *x* is between $-\varepsilon$ and $+\varepsilon$, then $|x| < \varepsilon$. The alternative argument included an error in the order of quantifiers by assuming that *x* can be chosen based on the value of ε, which is not the case in the original statement. The instructor asked Sam to discuss his reasoning about the alternative argument and he responded that the alternative argument cannot be a valid argument for the given statement. When pressed to discuss his reasoning for how he could tell, Sam had difficulty in doing so. This difficulty caused the perturbation necessary for Sam to create models of both the given statement and the instructor’s alternative argument. After prompting from the instructor Sam presented his model of the given statement as there is “*one value of x, for which the value of |x| is always less than ε.*” His representation of the instructor’s alternative argument was that “*basically, you pick some value of ε and then it tells you for what values of x*, (that)$|x| < \varepsilon$.” Once he built these models, Sam resolved his perturbation. When prompted by the instructor, Sam was able to describe the difference between the two statements, in which the order of the quantifiers had an impact on the meaning of the statements. The instructor’s introduction of the alternate argument raised a cognitive conflict that was resolved by the student Sam, which indicates the instructor’s use of the alternative argument was an effective use of DA.

Case 4: DA in the Combinatorics Experiment - Tree Diagrams

In the combinatorics teaching experiment, the instructor, the first author of this paper, also often provided alternative arguments to the students for evaluation. One example where she did so was when Kate and Boris were asked to solve the following task, which is adapted from Batanero et al.'s (1997) questionnaire:

Situation: Four children: Alice, Bert, Carol, and Diana go to spend the night at their grandmother's home. She has two different rooms available (one on the ground floor and another upstairs) in which she could place all or some of the children to sleep.

Question: In how many different ways can the grandmother place the children in the two different rooms?"

Boris determined the answer to be 2^4, explaining that there were two rooms that the first person could go to, for each of those possibilities, there were two possibilities for where the second person could go, and so forth. Then the instructor provided the tree diagram shown in Figure 2 as a solution provided by a supposed former student, Annette. At first Kate was confused by the representation and stated, "*I don't even know what that means*." After examining the tree diagram for a while, Boris stated, "*So I guess it's like doing it per person.* […] *She [is] pulling it apart like one person at a time. For the first person, they can either go to the ground floor or the upper floor. So like, you hold one constant. Say the first goes to the ground floor.* […] *And then the next person could go to the ground floor or the upper floor. So then, they both go to the ground floor for those* […] *four possibilities* (points to the top four leaves of the tree). *After that point* (points to the vertex G G _ _) *they* [*the third person*] *can go to the ground floor or the upper floor. So if they go to the ground floor* […] *and again there are two more possibilities for each of those. So there's two more there*."

Boris had made a connection between Annette's solution and the idea of holding something constant. He was able to pull out the essence of Annette's argument and explain it in his own words. Following Boris' interpretation of Annette's solution, Kate immediately responded, "*so this is just a graphic representation of what you* [Boris] *were saying*." This indicates that Kate, as well, was able to grasp the essence of Annette's solution and connect it to Boris' original solution even though she originally experienced some perturbation and did not immediately understand the tree diagram. The instructor's intention in providing Annette's solution was to raise awareness of the existence of visual representations for their current ways of thinking. Since the students were successful in building connections between Annette's solution and Boris' original solution, therefore further developing their reasoning and understanding, we consider the instructor's introduction of Annette's solution to be an effective implementation of DA.

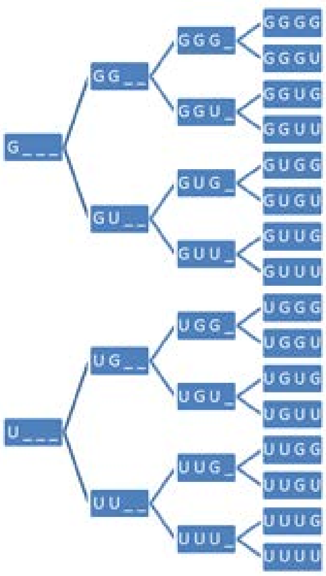

Figure 2: "Annette's Solution" provided through Devil's Advocate

Later in that session, Kate and Boris were attempting to determine the number of 3-letter "words" that could be formed from the letters *a*, *b*, *c*, *d*, *e*, and *f* if repetition of letters were allowed and the letter "*d*" must be used. They first over counted and found the answer to be $3\times6\times6$. The instructor provided a DA that determined the solution to be $6^3-5^3=91$. The students realized that both solutions could not be correct but they both had trouble identifying which solution was correct and which involved a flaw in reasoning. Boris and Kate used the tree-diagram in Figure 3 to solve the problem using a third method and confirm that the alternative solution provided was correct. Their tree-diagram differs vastly from the one supposedly written by Annette in the earlier task – the leaves in Figure 2 each represent an element of the solution set, but in Figure 3 all of the leaves are missing, many of the trees have only a root, and the use of slots to indicate where other items would be placed is inconsistent. However, Annette's idea of using a tree diagram to visually represent the elements being counted was adopted by the students. This seems to be an example of actor-oriented transfer (Lobato & Siebert, 2002).

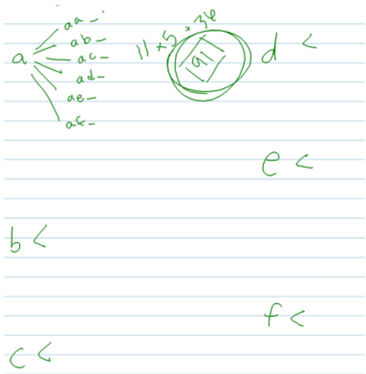

Figure 3: Kate and Boris's Transfer of Tree Diagrams

Discussion

We found that both Peer Interpretations (PI) and Devil's Advocate (DA) were effective instructional interventions designed to encourage students to critique the reasoning of others. Such activities challenged the students to understand the mathematics of a peer, fictional or otherwise, and provided opportunities to deepen conceptual understanding or reasoning. Indeed, these provocations could create sources of perturbation or assist in the resolution of such perturbation. In some cases, like in PI example from the real analysis study, the provocation may simply accomplish one of these tasks. In other cases, like both combinatorics examples and the DA example from real analysis, the provocation could both create the perturbation and help with its resolution.

This study has implications for both research methods and teaching practice. As shown, both DA and PI were effectively implemented in teaching experiments (Steffe & Thompson, 2000) and provided opportunities for further discourse, thus allowing the researchers to better understand the students' reasoning. We contend that DA could be implemented in clinical interviews (Clement, 2000) in a similar manner to help the researcher confirm his or her model of the student's mathematics. Because it is possible that a student's mathematics may change as a result of DA, we recommend the use of such provocation at the end of an interview. In a classroom, a teacher could implement either DA or PI to highlight differences in reasoning and raise or resolve cognitive conflict. In both cases, the reinterpretation by a student can include the student's own thinking and reasoning, while also including his or her own interpretation of the original argument. The interpreting student may adopt the meaningful aspects of the other argument into their own model of the situation. Indeed, we found evidence of this adoption in the student's assimilation of the idea to new situations in both combinatorics episodes discussed in this paper. Both DA and PI were effective in pushing students to further develop their reasoning by critiquing the reasoning of others.

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