PARTIAL UNPACKING AND INDIRECT PROOFS: A STUDY OF STUDENTS' PRODUCTIVE USE OF THE SYMBOLIC PROOF SCHEME

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This paper examines mathematics majors' evaluations of indirect proofs and of the compound statements used in this form of proof. Responses to survey items with a cohort of 23 students and six 1-hour clinical interviews, indicate that the students who could successfully evaluate indirect arguments and who could successfully recognize logically equivalent statements, tended to use partially unpacked (Selden & Selden, 1995) versions of the statements and the proofs and, in so doing, demonstrated a productive use of the symbolic proof scheme, whereas both successful and unsuccessful students tended to use a proof framework (Selden & Selden, 1995) for indirect proofs. Moreover, successful students' approaches are suggestive of activities, which are rarely found in introductory proof texts, yet may benefit novice proof writers.

Key words: Indirect Proof, Symbolic Proof Scheme, Proof Frameworks

Introduction

Research on students' production, understanding, and evaluation of indirect proofs suggests a general lack of preference for this form of proof. Harel and Sowder (1998) reported that students in their teaching experiments disliked indirect proofs and argued that this is due to students' preference for constructive proofs – proofs that construct mathematical objects and relations – over proofs that solely establish the logical necessity of a mathematical relation or object, as is the case with indirect proofs. Prior to Harel and Sowder's work, Leron (1985) also proposed that students' difficulties deriving a sense of conviction from indirect proofs may be rooted in the non-constructive nature of such proofs. Specifically, he argued that "most non-trivial proofs pivot around an act of construction – a construction of a new mathematical object," whereas with indirect proofs, we engage in acts of mathematical "destruction, not construction" (p. 323). Additionally, Harel and Sowder (2007), in their discussion of Aristotilian causality, have explored how students' tendency to view implications as causal statements can act as barrier to students' acceptance of indirect proofs. Indeed, if P → Q means P causes Q then how can ~Q → ~P show P causes Q?

Antonini and Mariotti (2008) explored students' proof preferences related to indirect proof and provided a rationale for these difficulties that is not solely rooted in issues of constructiveness (Leron, 1985; Harel & Sowder, 1998) and causality (Harel & Sowder, 2007). Working within the Cognitive Unity theoretical framework, Antonini and Mariotti showed that indirect proofs call on students to move from a principal statement (*e.g.*, P → Q) to a secondary statement (*e.g.,* in a proof by contraposition, ~Q → ~P) and to interpret or produce the proof of the secondary statement. They argue that it is this jump between principal statements (S) and secondary statements (S*) that is the source of students' difficulties and refer to such difficulties as *metatheoretical*. "Referring to their meta-theoretical status, we call the statement S*→S meta-statement, the proof of S*→S meta-proof, and the logical theory, in which the meta-proof makes sense, meta-theory" (p. 405). To illustrate students' metatheoretical difficulties, Antonini and Mariotti asked students to evaluate indirect proofs, including a proof by contraposition of the statement, "If n^2 is even then n is even." They showed that students struggle to accept the validity of the principal statement (*i.e.*, If n^2 is even then *n* is even), given the proof of the secondary statement (i.e., If *n* is odd, then n^2 is odd). For instance, Fabio, a university student remarked, "The problem is that in this way we proved that *n* is odd implies n^2 is odd, and I accept this; but I do not feel satisfied with the other one." (p. 407). We see here that Fabio has accepted that a claim has been made and a proof given of the secondary statement, but is not "satisfied" with regard to the principal statement. Thus, it is the "jump," S*→S, which he finds problematic. Interestingly, if one considers the ideas proposed by Leron (1985), and Harel and Sowder (1998, 2007), then a plausible rationale for the meta-theoretical difficulties discussed by Antonini and Mariotti (2008) emerges. Indeed, it may be that students look to such arguments to understand how S* *causes* S (*i.e.*, they seek causality in the argument) or how negating a claim of nonexistence (assuming a negation) enables one to *know*, in an epistemological sense, existence, that is, S* *means* S.

Reframing the Problem

According to Antonini and Mariotti, the Italian university students in their case studies recognized the transition to secondary statements (S*) and the proof of S* but experienced metatheoretical difficulties related to the "jump" between statements (*i.e.*, S*→S). It is unclear, however, if the majority of students would be as successful at recognizing and understanding this jump. Certainly, if one does not recognize and understand the jump between a principal and secondary statement, then one cannot accurately evaluate the logical structure of an indirect proof. Moreover, it may be the case that students' difficulties *accepting* the jump do not arise until and unless they are able to *recognize* a jump.

Drawing on series of clinical interview, Brown (2012) explored the extent to which advanced mathematics students (students enrolled in their 3rd and 4th university year, $n = 6$) were able to explain the logical structure of four indirect proofs involving basic number theory statements. Findings from this study indicated that only two of the advanced students could successfully explain the indirect proofs that were proofs by contradiction, while all but one student could successfully explain the indirect proofs that were proofs by contraposition. Additionally, the students who were successful at explaining the proofs by contradiction were also able to successfully evaluate a series of theorem statements, in terms of their logical equivalence, whereas the other four students struggled to do so. Specifically, among the other four students, after several unsuccessful attempts and repeatedly expressing doubts regarding the validity of their own evaluations of the argument, two were eventually able to explain the logical structure of the proofs by contradiction. The other two of the four students were not able to successfully explain the arguments even after repeated attempts and both misinterpreted key aspects of the logical structure of the arguments, including but not limited to the logical form of the secondary statement. Findings from these clinical interviews point to the possibility that many students' difficulties may be at the level of *recognizing* and *understanding* the jump between statements (S*→S). Moreover, they also suggest that the various forms of indirect proof (proof by contradiction, proof by contraposition) are not uniformly difficult for students.

As illustrated thus far, much of the research on indirect proof has focused on either students' lack of preference for or their difficulties with indirect proof (Leron, 1985, Harel & Sowder, 1998, Antonini & Mariotti, 2008). While understanding the nature of students' difficulties and the junctures at which these difficulties may manifest themselves is important, it is also possible that progress may be made by studying students who can (1) successfully interpret and evaluate indirect proofs, and (2) determine the logical equivalence of principal and secondary statements. The purpose of this paper is to share findings related to students' successful approaches related to interpreting indirect proofs. In particular, building on Selden and Selden's (1995) description of *unpacking* and their construct of a *proof framework* and Harel and Sowder's (1998) construct of a *symbolic proof scheme*, we will demonstrate that the students' successful attempts represent instances of students' use of a partial unpacking of the theorems and are examples of students' productive use of the symbolic proof scheme, while use of a proof framework for indirect proof was a characteristic of both successful and unsuccessful attempts. Interpreted in terms of the work of Antonini and Mariotti (2008), these findings shed light on the approaches used by students who can successfully engage in metatheoretical modes of thought.

The Study

The data presented in the paper are drawn from the *Bridges to Advanced Mathematics* study, which aims to identify content specific barriers to students' transition to advanced mathematics. One component of this study was a small-scale exploration of students' proof preferences, as they related to indirect proof. This exploration involved developing and administering an 8-item proof preference survey, which was administered to 15 students enrolled in courses typically taken by 3rd and 4th year mathematics majors (*e.g*., Topology, Analysis) and 8 students during the last week of an introduction to proof course in which more than half of students were in either their 3rd or 4th year. The survey instrument included three types of proof comparison tasks and two 'proof-related' tasks. Proof comparison tasks provided students with two proofs and asked the students to rate the extent to which they were confident in their understanding of each proof and to indicate which proof "is the most convincing" and which "is the best proof?" Three forms of proof comparisons items were used in the survey. The items ask the participants to compare: (1) a direct proof to an indirect proof (Type I); (2) a *Constructive* to a *Non-constructive Existence Proof* (Type II); and (3) a proof by contraposition to a proof by contradiction (Type III). Type III items were included to gather data on the question of whether or not there might be psychological distinctions to be made between these two forms of indirect proof. Type IV survey items were 'proof-related' comparison tasks, which asked participants to select a statement to prove out of three statements. Choices for the three statements include a principal statement ($\forall n, P \Rightarrow Q$) and two secondary statements. Secondary statements were either of the logical form $\forall n, \sim Q \Rightarrow \sim P$, which is the Contra-P form, or of the form "there exists no *n* such that, $P \wedge \sim Q$," which is the Contra-D form.

Following the administration of the surveys, 6 video-recorded, one-hour clinical interviews were conducted. During the interviews, participants were asked to discuss their responses to three of the proof comparison tasks and one statement selection task. The interviews were semi-structured to allow for clarification questions. The questions used in the interviews included asking the student to explain: (1) each proof to the interviewer (Can you explain this argument to me?); (2) any similarities or differences between the two arguments (Do you see any similarities or differences in the two arguments?) (3) his or her selection of the most convincing argument and of the best proof (Can you share with me how you thought about the two proofs as you decided which was more convincing? Can you share with me how you thought about the two proofs as you decided which was the best proof?). Furthermore, if the student did not comment on the proof type, students were asked at the end of the discussion of a comparison task, "would you describe one or either of the proofs as being a particular type of proof?" Thought of in terms of the work of Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2012), the questions asked of participants were primarily *local* comprehension questions, which they describe as questions focused on: "students' understanding of key terms and statements in the proof;" "students' knowledge of the logical status of statements in the proof and the logical relationship between these statements and the statement being proven;" and, "students' comprehension of how each assertion in the proof follows from previous statements in the proof and other proven or assumed statements" (p. 15). Additionally, some of the questions could be considered *holistic* - focused on students' understanding of the "proof as a whole" (p. 15).

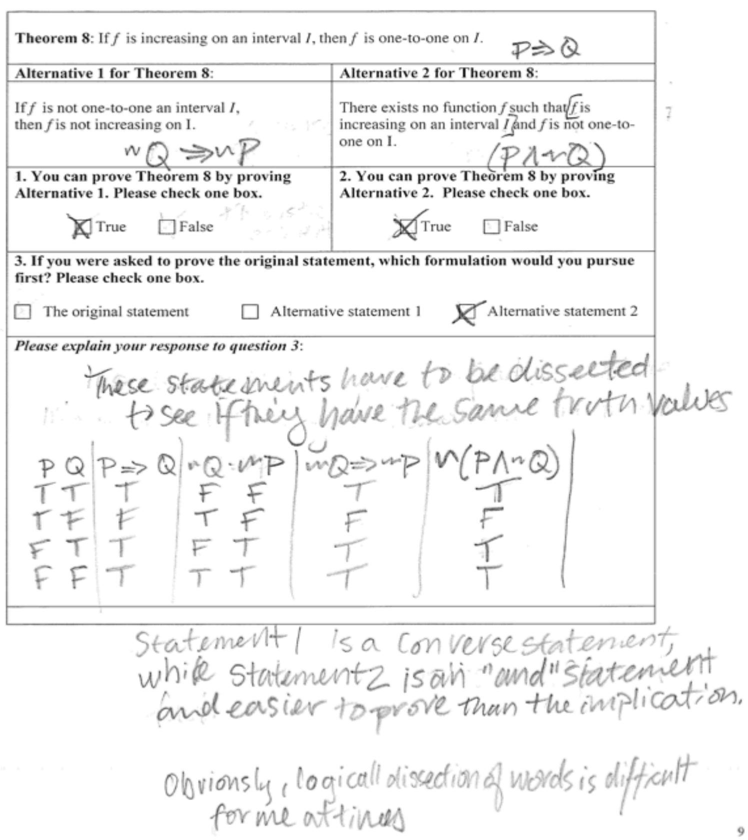
It should be noted that in Mejia-Ramos et al., the taxonomy of comprehension questions is geared towards evaluating students' comprehension of a single proof rather than a pairing of two proofs. Thus, this study's comparative questions (i.e., those focused on similarities and differences) do not fit within their taxonomy of comprehension questions. However, it can be argued that asking students to engage in comparative acts may provide additional insights into their comprehension of a given pair of arguments. Specifically, features that students may feel are not noteworthy may be important to making distinctions between two arguments. Indeed, comparative tasks may provide a context for eliciting a richer model of students' understanding of a given argument. Nevertheless, if such activities were non-normative then one could argue that it would be unlikely to provide deeper insight since student may be unprepared to engage in comparative work. Yet, it seems unlikely that this is the case. Students often engage in such activities. For instance, having constructed a proof for a theorem a student may compare their proof to an alternative proof provided by a teacher, a classmate, or in a text. One final issue to consider is that comparative questions may draw attention to specific features while diminishing others, a potentially problematic aspect of such questions. For instance, a comparison between two indirect proofs, such as in Tall (1979), might draw attention to specific details of the arguments, whereas comparison between an indirect and direct proof may draw attention to the indirect nature of an argument. In the study reported, however, understanding students' interpretations of the structure of indirect arguments was a primary research goal. Thus, the inclusion of comparative questions was warranted due to their potential focusing effect.

Analytic Approach

The analysis of the video-recorded interview data was informed by two constructs developed by Selden and Selden (1995): *unpacking* and *proof framework*. *Unpacking* refers to unpacking the logical structure of a statement; that is, the development of a symbolic, set-theoretic statement from a statement written in words. For example, the statement, "A function f is increasing on an interval I means that for any numbers x_1 and x_2 in I, if $x_1 < x_2$ then $f(x_1) < f(x_2)$" could be unpacked as, "$(\forall f \in \mathbf{F})(\forall I \in \mathbf{I})[(\text{f increasing on } I) \Leftrightarrow (\forall\, x_1 \in I)(\forall\, x_2 \in I)\ \{((x_1 < x_2) \Rightarrow f(x_1) < f(x_2))\}]$" (Selden & Selden, 1995, p. 138). The notion of *unpacking* was incorporated into the analyses due to observations of students' use of symbolic statements during the clinical interviews and in written survey work. A *proof framework* is a "representation of the 'top-level' logical structure of a proof, which does not depend on detailed knowledge of the relevant mathematical concepts" (p. 129). The construct of a proof framework was relevant to the analysis since students were asked both to explain the proofs and to describe the proof type, questions which might provoke a students' proof framework. Lastly, since some interview questions focused on the extent to which the students' found a particular argument convincing, the construct of a *proof scheme* – "what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244) – informed the analysis.

Findings

The findings reported in this paper focus on students' successful attempts interpreting and evaluating indirect proofs and the logical equivalence of secondary statements. Attention was drawn to this aspect of the data for two reasons. First, when asked to explain the indirect arguments many students experienced difficulties with the proofs by contradiction, which drew attention to the few successful students. Second, roughly 1/3 of the students did not successfully identify the contra-D form statements as equivalent in the Type IV survey items. Among the successful attempts was Anna's response, which is shown in Figure 1. This response pointed to the possibility that successful students may use a partial unpacking of the theorems and their proofs when determining logical equivalence and/or structure. *Partial unpacking* refers to the use of symbolic statements that are not fully quantified and would not be considered an *unpacking* as defined by Selden and Selden (1995) but do represent a movement from written words towards a symbolic form, hence the name *partial unpacking*. Observe that in Figure 1 the student has noted an implication ($P \Rightarrow Q$), then identified Alternative 1 as the contrapositive ($\sim Q \Rightarrow \sim P$) and employed a truth table in an effort to determine the logical equivalence, $(P \Rightarrow Q) \equiv \sim (P \wedge \sim Q)$. The student also noted, "Obviously, logical dissection of words is difficult for me at times." This remark points to the student's need to move away from a word-based language to a symbolic language in order to analyze the underlying logical structure. Thus, through a partial unpacking and a series of symbolic manipulations the student appears to have ascertained the equivalence of the statements in question; in other words, it appears that the student was "thinking of the symbols as though they possess a life of their own without reference to their possible functional or quantitative reference" (p. 250), in other words, a symbolic proof scheme was enacted.


Theorem 8: If *f* is increasing on an interval *I*, then *f* is one-to-one on *I*. P⇒Q

Alternative 1 for Theorem 8:	**Alternative 2 for Theorem 8**:
If *f* is not one-to-one an interval *I*, then *f* is not increasing on I. ~Q ⇒ ~P	There exists no function *f* such that *f* is increasing on an interval *I* and *f* is not one-to-one on I. (P∧~Q)
1. You can prove Theorem 8 by proving Alternative 1. Please check one box. ☒ True ☐ False	**2. You can prove Theorem 8 by proving Alternative 2. Please check one box.** ☒ True ☐ False

3. If you were asked to prove the original statement, which formulation would you pursue first? Please check one box.

☐ The original statement ☐ Alternative statement 1 ☒ Alternative statement 2

Please explain your response to question 3:

These statements have to be dissected to see if they have the same truth values

P	Q	P⇒Q	~Q·~P	~Q⇒~P	~(P∧~Q)
T	T	T	F F	T	T
T	F	F	T F	F	F
F	T	T	F T	T	T
F	F	T	T T	T	T

Statement 1 is a converse statement, while Statement 2 is an "and" statement and easier to prove than the implication.

Obviously, logical dissection of words is difficult for me at times

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Figure 1. Anna's Survey Response

The Anna's response was an anomaly among the written survey responses. It was not until the clinical interviews that reasoning similar to that in Figure 1 was observed. Due to space limitation, in this paper we will describe only one of the two successful cases.

Lillian was a sophomore mathematics major at the time of the interview. She was observed reasoning symbolically during a proof comparison task and a statement equivalence task. The proof comparison task is shown in Appendix A. When asked to describe the two arguments, Lillian immediately described Argument A as a proof of the contrapositive and then, after a period of hesitation, described Argument B as a direct proof. She was then asked to explain how Argument B was a direct proof. Proceeding, Lillian was observed rereading Argument B at least four times, repeatedly returning to the first sentence, and underlining the two assumptions stated. After approximately 90 seconds she had not yet responded to the interviewer's question. The interviewer proceeded by asking Lillian why she had repeatedly underlined "*x* and *y* have opposite parity." She responded by saying "it's not correct" and then "I need to read it again." After another 45 seconds, Lillian sat back from the paper and remarked, "Oh, I see … it's correct" and then noted, "the conclusion is contrapositive, umm, contradiction to what they assumed." Analysis of her scratch paper, showed two statements, P→Q and ~Q→~P, both of which she had gestured towards during her reading of the arguments, most often crossing over the latter statement – a gesture one could interpret as a "crossing out." At this point in the interview it was unclear how Lillian had arrived at the conclusion that the proof was valid. However, what was clear from her remarks, gestures, and markings was that she had struggled interpreting Argument B, had recognized the use of two assumptions, one of which was a negation of the conclusion of Theorem 6, and that she knew the basic structure of the arguments was not of the form ~Q→~P. With that said, deeper insight into Lillian's reasoning was obtained from her response to the Theorem 7 statement equivalence task (see Figure 3). As was the case with the previous task, Lillian immediately identified Alternative 1 as a statement of the contraposition of the original theorem. When asked to explain, she labeled "*n* is not a perfect square" as Q and "*n*(mod 3)≡ 2" as P and explained that Alternative 1 was of the form ~Q→~P.

Theorem 7: For all positive integers n, if n mod(3) = 2, then n is not a perfect square.	
Alternative 1 for Theorem 7:	**Alternative 2 for Theorem 7**:
For all positive integers n, if n is a perfect square, then n mod(3) ≠ 2.	There exists no positive integer n such that n mod(3) = 2 and n is a perfect square.

She then proceeded to examine Alternative 2 by rereading the statement multiple times and making a series of markings on her scratch paper, which are shown in Figure 4. She then proceeded to explain that the original statement was "for all" and that you could prove for all statements by, "showing that there exists no n such that it's … it's statement is false," she then gestured to her written work (the vertical arrow, phase 3) and noted that "these are equivalent," in reference to the statements P→Q and ~P ∨ Q, "so, this (points to P∧~Q) is the negation."

P→Q	P→Q ↕ ~P ∨ Q	P→Q ↕ ~(~P ∨ Q) ≡ (P∧~Q)	P→Q ↕ ~(~P ∨ Q) ≡ (P∧~Q) no n | P∧~ Q
Phase 1	Phase 2	Phase 3	Phase 4

Figure 4. Phases of Lillian's Written Work

Lillian's response resembles that shown in Figure 1 in that she produced a partial unpacking and then worked within the symbolic statements to determine equivalence, without any observable reference to the meaning of the statements, other than to confirm their logical status (*e.g.*, that one statement was the negation of another). Thus, ascertainment of the validity of an equivalence occurred at a symbolic level through the students' production of a partial unpacking and series of symbolic manipulations. Returning to Lillian's response to Theorem 6, her work with Theorem 7 offers a plausible rationale for her sudden realization of the structure of Argument B. Early in her work on Theorem 6, Lillian produced two symbolic statements $P \rightarrow Q$ and $\sim Q \rightarrow \sim P$, and reread Argument B repeatedly. Her crossing-out gestures indicate that she repeatedly rejected the idea Argument B was of either form. Thus, though it is unclear how she was able to move past her rejection of the argument and come to recognize it as disproving rather than proving $P \wedge \sim Q$, it is clear that a set of partially unpacked statements were used as tools for examining the logical structure of the argument. Moreover, with both tasks, Lillian constructed a series of partially unpacked statements and reasoned with those symbols as though "they have a life of their own" – in other words, she used a symbolic proof scheme.

It is also the case that Lillian and many other students recognized that within Argument B (and the other indirect arguments) a contradiction occurred and then used this realization to discuss the structure of the proof by contradiction. When asked to explain how the argument related to the theorem, however, many students became confused and were unable to make connections between the theorem statement and the basic assumptions in the initial sentence of the argument. Thus, it appears that a rudimentary proof framework for proof by contradiction was invoked among all of the students but that this was not a characteristic that distinguished successful from unsuccessful students.

Concluding Remarks

Researchers, such as Harel and Sowder (1998) and Healy and Hoyles (2000), have provided many examples of students' unproductive use of a symbolic proof scheme in their research. Few if any researchers have provided evidence of productive uses – of what Harel et al. alluded to when they remarked, " symbolic reasoning can either be superficial and mathematically vacuous, or a very powerful technique" (p.250). The findings in this paper, however, highlight the possibility of a potentially productive use of the symbolic proof scheme – one that may aid students in their effort to navigate the logic complexities and metatheoretical issues of indirect proofs. Moreover, the successful students' approaches are suggestive of activities, such as partially unpacking, which are rarely found in proof texts, yet may benefit students in their efforts to understand the structure of indirect proofs.

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Appendix A

<table>
<tr><td colspan="2">Theorem 6: If x and y are two integers for which x + y is even,
then x and y have the same parity.[1]</td></tr>
<tr><td>Argument A:

Assume x and y have opposite parity. Since one of these integers is even and the other odd, there is no loss of generality to suppose x is even and y is odd. Thus, there are integers k and m for which x = 2k and y = 2m+1. Now then, we compute the sum x + y = 2k + 2m + 1 = 2(k+m) + 1, which is an odd integer by definition.</td><td>Argument B:

Assume x and y are two integers for which x + y is even and that x and y have opposite parity. Since one of these integers is even and the other odd, there is no loss of generality to suppose x is even and y is odd. Now then, we compute the sum x + y = 2k + 2m + 1 = 2(k+m) + 1, which is an odd integer by definition. However, by assumption x and y are two integers for which x + y is even. Since x + y cannot be odd and even, either x and y must have the same parity or x + y is not even.</td></tr>
<tr><td>1. I am confident about my understanding of Argument A. (Please mark one)

☐ Strongly agree ☐ Agree ☐ Disagree ☐ Strongly disagree</td><td>2. I am confident about my understanding of Argument B. (Please mark one)

☐ Strongly agree ☐ Agree ☐ Disagree ☐ Strongly disagree</td></tr>
<tr><td colspan="2">3. Which argument, in your opinion, is the most convincing? ☐ Argument A ☐ Argument B

Please explain your selection. (If you need additional space please use the back of this page.)</td></tr>
<tr><td colspan="2">4. Which argument, in your opinion, is the best proof? ☐ Argument A ☐ Argument B</td></tr>
</table>

1. Two integers are said to have the same **parity** if they are both odd or both even.