BRINGING THE FAMILIAR TO THE UNFAMILIAR: THE USE OF KNOWLEDGE FROM DIFFERENT DOMAINS IN THE PROVING PROCESS

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This report considers student proof construction in small groups within an inquiry-orientated abstract algebra classroom. During an initial analysis, several cases emerged where students used familiar knowledge from another mathematical domain to provide informal intuition. I will report on two episodes in order to illustrate how this intuition could potentially aid or hinder the construction of a valid proof.

Key words: [Abstract Algebra, Proofs]

A great deal of attention has been given to student construction of proof particularly in an abstract algebra setting (Weber & Larsen, 2008). Hazzan (2001) has observed that undergraduates' difficulties in group theory may partially be attributed to the abstraction level. Both Hazzan (2001) and Selden and Selden (1987) found that students often retreat to familiar number systems when working on algebra proofs.

As part of a larger project implementing a group theory curriculum based on guided reinvention of group and isomorphism concepts, this study considers student proof construction in small groups. Through an initial analysis, several cases emerged where students used familiar knowledge from another mathematical domain to provide informal intuition for an argument. This divergence from abstract algebra is consistent with the return to familiar contexts seen with Hazzan (2001) and Selden and Selden (1987) but unfolds in a different manner. In this proposal, I will present two such cases where students utilized prior knowledge in an unexpected way during the informal stage of their proof development.

Theoretical Background

Zandieh, Larsen and Nunley (2008) discussed the role of student intuition as students moved from informal notions to formal proofs. The researchers suggested, "As students search for key ideas and work to relate a key idea to the arguments needed to provide a rigorous proof, they often need to develop an intuitive sense of how the system in question works" (p. 122). The key idea connects intuition with a formal proof (Raman, 2003). Referencing Fischbein (1982), Zandieh, Larsen and Nunley (2008) categorized two types of intuition in proof: affirmatory and anticipatory. The former can be coercive in that "an individual with this intuition may not be able to consider other alternatives" (p. 126). Whereas, anticipatory intuition is associated with a feeling of certitude, but the "intuitions anticipate a further refinement" into formal proofs (p. 126).

I will consider two episodes where informal notions emerge when students bring prior mathematical knowledge into the abstract algebra context. The first example will illustrate affirmatory intuition whereas the second will illustrate anticipatory intuition.

Research Methodology and Problem Context

Four implementations of an inquiry-based Algebra course were videotaped over the course of two years. This course largely served as an introduction to group theory where many students were engaged in proving for the first time. Video data from these implementations were analyzed in this report. An initial search served to isolate the portions of the curriculum where students were prompted to prove in a small group setting. During this process, I noticed several occurrences of students referring to mathematical contexts outside of abstract algebra while in the early stages of constructing a proof. After consulting the literature, these videos were analyzed again with attempts to categorize the intuitions as affirmatory or anticipatory.

I will consider episodes from two classrooms implementing a curriculum where formal abstract algebra concepts are developed using students' informal knowledge. At this stage, the students have developed a familiarity with the symmetry group of an equilateral triangle. They have developed the symbols in terms of R and F to represent a 120 degree rotation and a vertical reflection respectively. Through their investigations they've developed a list of rules that describe the composition of symmetries (see Figure 1). Typically, the existence of inverses are not included. For more details of this curriculum see Larsen (2012).

Rules

The associative property: If A, B, and C are symmetries then $A(BC)=(AB)C=ABC$.

The identity property: Any symmetry combined with I is that symmetry. If X is a symmetry then $IX = X$.

$RRR=I$ and $FF = I$.

$RF = FRR$.

Figure 1. Student Rules for Triangle Symmetries.

While creating a table for the symmetries, students notice the pattern that each symmetry appears exactly once in each row. In order to motivate the idea of inverse elements, the prompt in Figure 2 was given. The students worked on proving both parts in small groups.

Conjecture: Each symmetry appears exactly once in each row of the table.

Can we prove this conjecture using our rules? If not, can we come up with another rule to add to the list that will allow us to prove the conjecture?

Hint: This conjecture can be broken down into two parts, work on one at a time.

1. Each symmetry appears at most once in each row.
2. Each symmetry appears at least once in each row.

Figure 2. Prompt

Results

The following cases illustrate students making connections to familiar mathematical contexts when beginning the informal process of proving the conjectures above. Each case comes from a different class where groups of four students worked together to attempt to prove the statements. The first case serves as an example where the intuition did not lead to the construction of a proof.

The second case illustrates how a student using prior knowledge provides the intuition for the key idea of inverses.

Case 1: Prime Numbers

In this first case, Bob began the discussion by introducing an analogy to prime numbers stating, "It's almost like prime numbers. You can have the composition of all of them but they trickle down to a single action. And it's 8 distinct actions, or I'm sorry not 8, 6 distinct positions [referring to the six symmetries of the triangle] in terms of orientation and location." He continues his analogy, "Kind of like in the sense of a prime number, you can do all sorts of things to it to make it look different, when you factor it out into a prime you can't go any further. You can't split it any further without breaking it into a decimal." Bob is attempting to connect his ideas of symmetries with his knowledge of prime number factorization.

Bob and his group-mates continue the conversation with reference to this simplification idea.

Roger: I see the spirit of what you are talking about and it's something in that uniqueness that you get that if you don't have ...if there are no repeats in the rows and columns that you are going to compose, then you can't get a repeat in that row or that column.

Bob: Because it always factors down to something that's already there.

At this point Roger and Bob are attempting to coordinate the connection to prime numbers with the uniqueness of each element in a row. This connection appears backwards since they were considering the multiplication of symmetries as opposed to factoring. They may have been confounding the idea of closure (that any combination of symmetries reduces to a known symmetry) with the property of appearing exactly once in each row. Alternatively, they may be thinking of one of the factors being held constant acting the same as a symmetry multiplied on a given row. This latter interpretation would align with the statements that follow. After the prime number discussion, Winston presented the following proof:

Winston: Since there are six unique functions and in each row and column these functions are composed with exactly one unique function that row or column must contain six unique functions because they all are equal to the same function.

Gayle: They are all unique so anything you do with them is going to be unique.

I would hypothesize that this prime number metaphor was serving an affirmatory role. All proof attempts resembled Winston's above where the uniqueness served as the reasoning. None of the group members considered justifying further. Instead, the intuition acted as a hindrance to the construction of a valid proof.

Case 2: Invertible Functions

In this second case, Logan contributed knowledge of invertible functions to make an argument for the conjecture:

Logan: Each of the operations is invertible because for any of the operations like R there is something you can compose with that to turn it into the identity. So if you have R and R squared and put them together, then you get the identity. And since it's invertible then it must be, um, one to one. I don't remember [inaudible.] Since it's invertible the same entry can never appear in the same row-

Henry: How does it being invertible prove that?

Logan: Because say you take FR and what's the inversion of that $(FR)^{-1}$. Actually, FR is the inversion of itself. So you have a bunch of starting points and you are mapping using a function FR and we know that if we do FR again, then it's going to map *them all back to the same place, so we know that we can't ever map two to the same one or we wouldn't be able to map it back. Because we know that FR FR equals I. So that means for any input rotation FR. For any two input rotations into FR, you can't get the same output rotation* (see Figure 3).

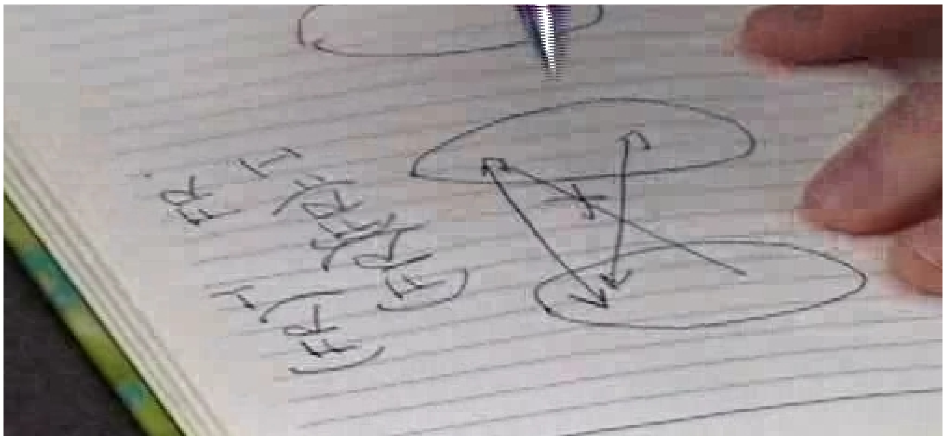

Figure 3. Logan's Function Diagram for the inverse of FR

Logan went on to conclude that "For each of them there is an inverse. Because they are invertible, they all have to be 1-1 and onto." As evidenced by the diagram in Figure 3, Logan was able to build an intuition about the use of inverses. I would conjecture that this knowledge came from precalculus or other study of functions based on his use of a function diagram. Logan was identifying symmetries with functions mapping the group onto itself.

I would argue that this connection to invertible functions served in more of an anticipatory role. Logan used his function diagram and connected it to each symmetry having an inverse. The anticipatory intuition was further evidenced later in the conversation as the group attempted to formalize the idea culminating in Henry suggesting a proof by contradiction with direct reference to Logan's inverse suggestion.

Discussion

These episodes demonstrate two very different paths into a familiar domain. In the first case, the student group created an analogy to prime numbers that ultimately left them unable to prove the conjecture. In the second case, the introduction of invertible functions provided the foundation for the key idea of inverses. Further research into the types of connections made by novice provers could help inform instruction in order to encourage students to develop the intuition necessary to become successful in proof construction.

Questions for the Audience

1. What role can instructors play to encourage the formation of anticipatory intuition? Likewise, how can instructors help students move beyond affirmatory intuitions?
2. Have you noticed students appealing to prior mathematical knowledge during the intuition stage of proving?

References

Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics*. 3(2), 9-24.

Hazzan, O. (2001). Reducing abstraction levels when learning abstract algebra concepts. *Educational Studies in Mathematics*, 40, 71-90.

Larsen, S. (2012). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *Submitted to the Journal of Mathematical Behavior.*

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics* 52, 635-638.

Selden, A. and Seldon, J. *(1987).* Errors and misconceptions in college level theorem proving. *In Proceedings of the 2nd International Seminar on Misconceptions and Educational Strategies in Science and Mathematics.* Cornell University, NY.

Weber, K. & Larsen, S. (2008). Teaching and learning group theory. In M. P. Carlson & C. Rasmussen (Eds.) *Making the connection: Research and practice in undergraduate mathematics* (pp. 137-149). Washington, DC: Mathematical Association of America.

Zandieh, M., Larsen, S., and Nunley, D. (2008). Proving starting from informal notions of symmetry and transformations. In M. Carlson & C. Rasmussen (Eds.) *Making the Connection: Research and Teaching in Undergraduate Mathematics* (pp. 123-135). Mathematical Association of America