# INVESTIGATING STUDENT UNDERSTANDING OF EIGENTHEORY IN QUANTUM MECHANICS

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*An initial investigation into students' understanding of Eigentheory using semi-structured interviews was conducted with students at the end of a first-semester course in quantum mechanics. Many physics faculty would expect students to have mastery of basic matrix multiplication after a course in Linear Algebra, and especially so after fairly extensive use of matrices in quantum mechanics in the context of Ising model spin problems. Using a previously published interview protocol by Henderson et al, student reasoning patterns were investigated to probe to what extent there reasoning patterns were similar to those identified among Linear Algebra students. Reasoning patterns appeared quite consistent with previous work; that is, students used superficial algebraic cancellation, and demonstrated difficulty interpreting their result even when they arrived at a correct solution. The interview protocol was modified slightly to probe whether or not students felt the tasks they were engaging in were mathematical or physics-related. Additional questions were added at the end of the protocol about how these concepts were used in their quantum mechanics course. Students were somewhat successful relating them to Hamiltonians and energy eigenvalues, but couldn't articulate the type of physical situations where they might be useful.*

*Key words:* Linear Algebra, Physics, Quantum Mechanics, Interdisciplinary, Interviews

Over the past decades, a great deal of research has taken place within both Mathematics Departments and Physics Departments on the learning and teaching of these respective fields. There is a small, but growing field of researchers that are interested in tapping the knowledge base of both the communities of Physics Education Research (PER) and Research in Undergraduate Mathematics Education (RUME). This work was inspired by a paper at the 2010 RUME Conference, which shed light on students' reasoning on concepts of Linear Algebra (Henderson et al. 2010). The study focuses on students' understanding on matrix multiplication and geometric interpretations of these operations.

These principles are mathematical concepts that physics students use throughout their upper-division courses and into graduate studies in courses such as Quantum Mechanics (QM), Particle Physics, Electricity and Magnetism, and Mechanics. Curricula in undergraduate QM courses in particular require students to apply these mathematical concepts to identify "energy" eigenstates and eigenvalues for a particular system that is defined by a Hamiltonian operator. After completing instruction in a QM class, instructors expect students to be well versed in using these concepts to solve and interpret physical situations.

There are only a handful of investigations on students' use of mathematics at the upper-division in physics. (Pepper et al. 2012, Bucy, Thompson et al. 2006, Bucy et al. 2007) The initial phase of this study is to investigate how QM students respond to this interview protocol and get a sense of their reasoning patterns. All three students had previously taken Linear Algebra, and displayed a range of thinking about eigenvalues, eigenfunctions and operators. All three students were very proficient at "doing the math" in as much as determining the equations they need to solve to determine the eigenfunction, but were uneven in their description of what the results meant.

## Framework

The theoretical framework of this work is identifying student difficulties by Heron (2004). This framework has an explicit goal of identifying things that students know well as well as ideas with which they struggle. Its design is meant as the first step in a broader research agenda of developing curricular materials that explicitly build on correct student conceptions and target student difficulties though instructional interventions. It assumes that student responses (be they written, spoken and otherwise) are representative of their thinking about the question at hand. If student thinking is unclear to the interviewer, clarifying questions are asked until the interviewer feels they have been given the best explanation the student is likely to give.

## Methods

The interview protocol consists of several questions about interpreting matrix multiplication, and also doing a matrix multiplication (see Fig. 1 and 2). Students' demonstration of their mathematical reasoning was uneven throughout the interviews. One student in particular, demonstrated previously reported difficulties by treating the first problem in Figure 1 as a simple algebra problem, answering that A was equal to 2.

**Figure 1. The first prompt from the interview protocol by Henderson et al.**

*Consider a 2x2 matrix A and a vector* $\begin{bmatrix} x \\ y \end{bmatrix}$*. How do you think about* $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?

*How do you think about what the equals sign means when you see it written in the context of this equation?*

Student response: *"When I see the equals sign, I guess it tells me that everything on the left half is the same as everything on the right. Since I also see that the matrice [sic] on the left is equal to the matrice [sic] on the right. It lets me know that the only difference between the left half and the right half on the left is an A and right half is a 2. So I assume the A has to be 2. When I think about the equals sign and that's what it tells me."*

When describing what the equals sign means, the student uses what Henderson et al. called superficial algebraic cancelation. When given the values for the 2x2 matrix (in Fig. 2), he recognizes there is an issue with his previous work. Instead of wrestling with this inconsistency further, the student immediately begins plugging away.

Student response: *"Now I think about this expression… here [pointing to previous] we have a matrix and a constant and I said 'hmm, makes me think A was equals the constant'. Now we have matrix and a constant, and hmm, it doesn't quite sound equal. I know what I can do, I'll multiple this through and see what it looks like."*

The student continues until he has expressions for the values of x and y that make the expression true. There is an explicit prompt later in the protocol to do this, but the student does it all spontaneously.

**Figure 2. Prompt from the interview protocol.**

*Suppose* $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$*. Now how do you think about* $\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$?

The student is asked by the interviewer if any of these questions seem more like physics questions or more like math questions, and the student identifies the first question (Fig. 1) as being more like math and the second question (Fig. 2) as being "more like doing physics". This differentiation of knowledge on the part of the student is interesting because it implies that domains of mathematics knowledge and physics knowledge may be different for students. This also implies that students may be framing problems as being of a particular domain that may influence their ability to solve problems (Tuminaro and Redish 2007). The concept of framing ties into the theoretical framework of resources (Hammer and Elby 2003), that is common in physics education research. Resources have a great deal of overlap with diSessa's knowledge-in-pieces model of student thinking.

The general idea of epistemic framing within a resources framework means that if students view a problem to be from a particular domain, they will attempt to access bits (often called resources) of knowledge and procedure that have previously been helpful for answering questions within this domain. Students may have difficulty accessing knowledge they possess from a different domain (e.g., physics) when they frame a particular problem to not include that domain but rather something else (e.g., mathematics). In the above scenario, the student identifies a strategy that they feel can work for them ("multiple this through and see what it looks like"), and later identifies this as "doing physics" when in fact there is no physics context whatsoever.

After the episode where the student demonstrates some understanding about what can be done when giving a matrix operator, the student attempts to reconcile the previous answer.

Student response: *"All of a sudden I think I would've figured out A… if I plug in 10, and -10 I get a new expression for A, but now that's dead in the road. I don't think it'll give me anything new."*

It seems that there is a disconnect between what the student initial identifies as correct and what their work in the second in the interview and the work from the first part of the interview. The student makes no further progress in understanding where the error from the first question lies.

## Future Work

Future work will attempt to further probe the boundary of students understanding of mathematics and physics concepts and how students attempt to use physic and mathematics ideas to answer questions in the complimentary domain. Given the piece-wise nature of student responses and the small sample-size of students in upper-division of physics, it seems that a more appropriate theoretical framework would be more suitable for making sense of student thinking. As this work progresses alternative frameworks will be considered in an attempt to cast additional light on students thinking.

## Questions

This work is in a very preliminary stage. It wasn't entirely clear that using a protocol from on Linear Algebra concepts would fly at all among physics students. A lot of questions remain for myself as a researcher about what I can do and what I should be doing with this work. I would like to use these first steps as a springboard for a refined protocol that draws out their mathematical knowledge, but also has them tackling explicit physics problems that use these mathematical tools. Could this be done in a single interview? Should these questions be administered over the course of several interviews? Perhaps it would be better to have them done in a group of students to capture more authentic dialogue? Might it be better to capture actual classroom dialogue or ask students to record their own discussions while doing homework or other assigned tasks outside of class?

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