ON THE ROLE OF PEDAGOGICAL CONTENT KNOWLEDGE IN TEACHERS' UNDERSTANDING OF COMMUTATIVITY AND ASSOCIATIVITY

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The purpose of this study is to investigate a relationship between mathematical content knowledge and pedagogical knowledge of content and students (Hill, Ball, & Shilling, 2008), in the context of algebra. As participants in a paired teaching experiment, mathematics education doctoral students revealed their understandings of commutativity and associativity (cf. Larsen, 2010). Although the participants' knowledge of children's initial understandings of algebra and familiarity with mathematics education literature influenced their own mathematics reasoning, the difficulties they encountered were similar to those of undergraduates without such pedagogical content knowledge.

Keywords: Abstract Algebra, Teaching Experiment, Pedagogical Content Knowledge

Introduction

The purpose of this study is to investigate the ways in which doctoral students' learning of introductory abstract algebra differs from that of undergraduate students. Though the mathematics course background of undergraduates and mathematics education doctoral students may be similar, doctoral students generally have greater study of pedagogy. Thus, implicit in this goal is an exploration of how pedagogical content knowledge (Shulman, 1986) on the part of the student affects the learning of mathematics content.

Background

Larsen (2009) articulated a curriculum for an inquiry-based undergraduate abstract algebra course that includes students' reconstruction of the group axioms, using the design constructs of guided reinvention (Gravemeijer & Doorman, 1999) and emergent models (Gravemeijer, 1999). Guided reinvention is an instructional technique by which students gain a sense of ownership of their understandings, in contrast with the more authoritarian lecture format that is more typical of undergraduate teaching. Whereas a traditional abstract algebra course may begin with an expressed definition of a group and use S_3 as an example, the first unit in Larsen's (2012) curriculum includes students' identification of the symmetries of an equilateral triangle and subsequent creation of their own symbol systems, notation, and organization structures. Thus, the instructional design begins with students' creation of models of their own mathematics experiences. As students reflect on these emergent models, their conjectures and attention become less focused on the specific actions that resulted in the models' creation, and, via teacher prompting, their abstractions can be guided toward a formalization of the group axioms (Larsen, 2009).

Larsen (2010) reported on the results of two single-session teaching experiments with pairs of undergraduate mathematics students, the purpose of which was to inform the instructional design for use in a whole-class setting. A main finding of the paired teaching experiments was undergraduates' difficulties with the associative and commutative properties. Upon consideration of the research literature showing similar findings involving children and teachers, Larsen (2010) concluded that such difficulties "may stem from 1) a tendency to think about expressions involving binary operations in terms of a sequential procedure and 2) a lack of preciseness in the informal language used in association with these properties" (p. 42).

Theoretical Framework

Hill, Ball, and Shilling's (2008) elaboration of Shulman's (1986) seminal work distinguishing the types of knowledge of content and pedagogy required of K-12 teachers is used to frame the discussion of the role that specialized knowledge for teaching plays in the learning of mathematics (see Figure 1). Hill et al. (2008) describe specialized content knowledge (SCK) as the knowledge of the mathematics that is particularly relevant for teaching, beyond the common content knowledge (CCK) that might be used by a more general population. They describe knowledge of content and students (KCS) to be the understanding of *how* students learn the content, while knowledge of content and teaching (KCT) is understanding of the effect of teacher actions on said learning. Knowledge at the mathematical horizon (KMH) is both an "advanced perspective on elementary knowledge" and an application of "advanced mathematical knowledge" to lower-level curricula (Mamolo & Zazkis, 2011, p. 4).

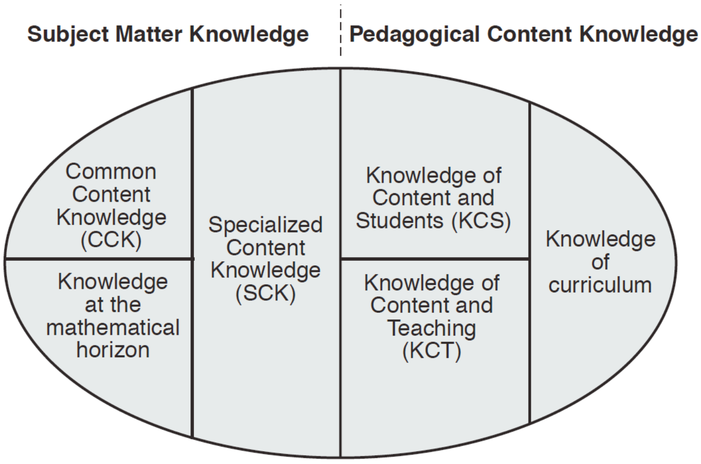

Figure 1. Hill et al. (2008) Domain Map of Knowledge for Mathematics Teaching

Hill et al. (2008) acknowledge that in their conceptualization, "KCS is distinct from teachers' subject matter knowledge. A teacher might have strong knowledge of the content itself but weak knowledge about how students learn the content or vice versa" (p. 7). The model has been extended to university level mathematics in studies of instructors' KCS (e.g., Alcock & Simpson, 2009; Johnson & Larsen, 2012; Speer & Wagner, 2009). The current study considers a reverse relationship – the effect of a *student's* pedagogical content knowledge on her *own* learning of university level mathematics content.

Participants

Both of the participants in this study, Zowie and Mary, were female second-year doctoral students in mathematics education at a large Southeastern University. Zowie and Mary each had over ten years of mathematics teaching experience; Zowie's was primarily with middle school geometry in China, while Mary's was with the elementary grades in the U.S. Though they each had some exposure to group theory from prior coursework, neither Zowie nor Mary recalled completing a university course devoted to abstract algebra. Both participants (and the researcher) were enrolled in a doctoral seminar that examined research in undergraduate mathematics education, and they were recruited because of their expressed desire to learn the mathematics content before digesting educational research about its teaching and learning.

Method

Larsen's (2012) instructional materials were used [with permission] to partially replicate his 2010 study; major differences included the participant population (mathematics education doctoral students instead of undergraduate math majors) and the number of sessions (four rather than one). The general trajectory of Larsen's (2012) group unit begins with students manipulating an equilateral triangle and developing notation for the compositions of its symmetries, writing the symmetries in terms of compositions of a single reflection and a single rotation, organizing them in a Cayley table, and constructing a minimal set of properties that could be used to perform the compositions. In the current study, the participants did not fully develop the notion of group by the end of their last session, so the researcher disclosed the intent of the instructional sequence and the relationship between the activities and the formal definition of a group at the end of the final session.

The researcher videotaped each session, and he watched the videos between sessions and read the online instructional material provided by Larsen (2012) to prepare materials for the following session. The role of the researcher throughout the first three hour-long sessions was that of a facilitator, as the participants generally questioned each other's thinking and asked for clarification without prompting. The researcher encouraged the participants to verbalize not only their own mathematics, but also their hypotheses for why they (or their partner) might reason a particular way. For example, before developing an initial notation system for the symmetries of an equilateral triangle, they were given definitions of isometry, symmetry, and equivalent symmetries. The participants negotiated the meanings for these definitions, and they subsequently verbalized their perceptions of the sources of their concept images (Vinner, 1991).

Results

Zowie and Mary developed their own (shared) notation for the six symmetries of the equilateral triangle (see Figure 2). They were tasked with writing each of the six symmetries in terms of the following: the 120-degree clockwise rotation (R) and the reflection about the vertical axis (F). Zowie and Mary developed additive notation, e.g., R + R + R = 3R, which was used throughout the sessions. Protocol 1 begins as they begin to complete a table of pair-wise combinations of symmetries and are deciding how to name the identity [S: Researcher, M: Mary, Z: Zowie].

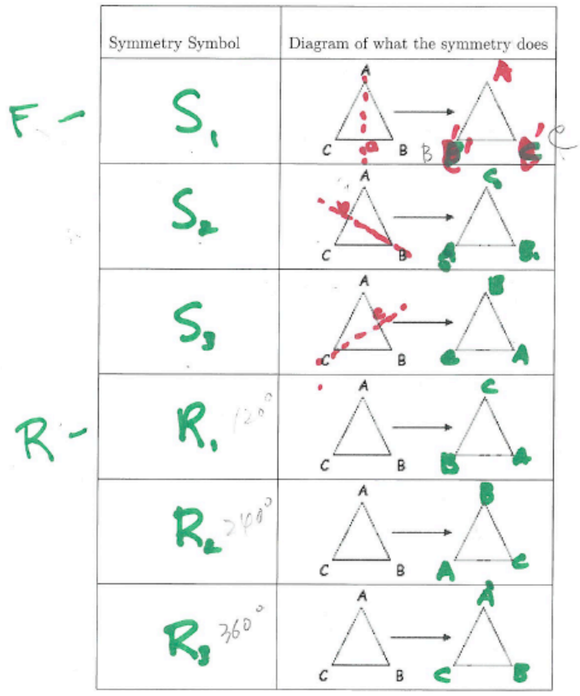

Figure 2. Zowie and Mary's Six Symmetries

Protocol 1:Session Two Discussion about commutativity

S: What is your identity the same as, in terms of Rs and Fs?

M: R_3.
Z: [at the same time] 3R.
M: 3R. [Writes 3R in the box corresponding to F+F].
Z: Okay, then we go, this [F] this [R + F]?
M: Um hmm. So two flips and one rotation.
Z: No. One flip, one ro[tation], and one flip [pointing at F, and then R, and then F]. I will go in this way.
M: You want to go in that order? Let me go in my order, and we'll see if we get the same thing. [Both participants complete the operations]
Z: 2R.
M: I got BCA. You got CBA? [Mary demonstrates what she did while Zowie watches, and then vice versa.]
M: Okay, so order matters.
Z: Yes, that's right. [Mary writes the rule "order matters" at the bottom of her paper.]
Z: It's the same as last time when you present…the equal sign is not truly equal.
M: Laughs. I have to know this – so we know that commutativity does not play a role.

Zowie's comment about the equal sign indicates that she was associating the use of equivalence classes of symmetries with a lack of commutativity. In the first session, she had expressed concern over whether three 120-degree rotations would be considered equivalent to one 360-degree rotation. When Zowie found that F + R + F did not result in the same symmetry as F + F + R, she returned to a conception that equivalent symmetries were not "truly equal." Her objection to Mary's suggestion to compute F + F was not because it would require changing the sequence of operations; she resisted replacing two actions with their composition.

After the participants completed the chart, they were asked what other rules or patterns they had noticed. The intent was for them to identify a minimal list of rules that would be necessary to complete the table. The rules the participants had formulated at this point in the teaching episode were: "3R = identity; 2F = identity; R + F = F + 2R; F + R = 2R + F; Order matters (commutativity does not work)". The next activity was to re-complete their Cayley table using only these rules – the original chart and triangles were removed from view. Protocol 2 describes how Zowie found some of the more complicated relationships on scratch paper using an associative property without initially acknowledging its use.

Protocol 2: Session Two Discussion about associativity
S: How did you get the identity here [2R + F + 2R + F = I]?
Z: I did it in two ways. I switch this [circles the first 2R + F] to make F + R + 2R + F. This [circles R + 2R] is 3R, which is the identity. So [we have] F + F, which is the identity. [Goes on to demonstrate that another way began with substituting for middle, F + 2R].
S: So in all these ways, it didn't matter if you substituted for this[underlined first 2R + F], this [underlined F + 2R] or this [underlined second 2R + F].
Z: Yes, because it's not order. I didn't switch the order. I just used a different combination.
S: And that's okay. Is that important? Should we write that down, is it obvious?
M: I think it's important.
S: How can we write that down?
M: That's in a way, saying that left to right is not important. How would you describe that? Order of operations, in a way? I'm trying to think of how to say that.
S: [3 second pause] So you didn't change the order that you worked. What you changed was, which operation you did first.

M: [Quietly, as if questioning] Like associative property?
S: Why do you say it's like associative property?
M: Like if you had parentheses around it, you could change the parentheses, which would indicate where you started first. What operation should begin, first.
Z: Wouldn't that be switching the order?
M: We're switching the order of the operation, not the order of the addends.
Z: Um hmm.
S: So you're still maintaining the pair, [pointing to 2R + F] in that order, you're not switching this to be F + 2R.
M: We're just deciding which operation to begin with, which is more associative.
Z: I didn't switch the order of addends, and I [stopped mid-sentence]
S: Right here, [pointing to paper] you did R + 2F first, even though F was written first.
M: It was like you had parentheses that grouped them, and instead of doing this, I'm going to put my parentheses around here and start here first. So that's how I look at it. The associative property focuses on which operation you're going to begin with, by using parentheses. Switching those parentheses is indicative of the associative property.
Z: I'm thinking, theoretically I only look at the operation – here [gestures to paper] it's right. But then you think back to actions – you're actually changing the order of the actions.
M: That's what an operation is, isn't it? The order of actions, isn't that the order of the operations? What's the difference?
Z: I should say it in this way. For this [indicates paper] I'm just playing a mathematics game. In here. But not the action.

Zowie's last comment is strikingly similar to a statement by Erika in Larsen (2010), who also referred to the use of the associative property as having " nothing to do with the actual order you're flipping the triangle in. Like it's all a paper game kinda" (p. 40). Mary's verbalization that "switching those parentheses is indicative of the associative property" is evidence of her thinking of how students understand associativity –as recognition of a change in grouping symbols' placement rather than a property of a binary operation. In the subsequent session, both Zowie and Mary declined to give up the rule that "order matters," a finding also reported by the undergraduates in Larsen (2010). Zowie and Mary each gave powerful statements demonstrating how their KCS influenced their reasoning.

Protocol 3: Session Three Discussion about the necessity of the "order matters" rule.
S: Why do we need to know that order does matter in order to fill out the chart?
M: Otherwise you're going to start to group your similar letters.
Z: Um hmm.
S: Why?
M: It's based on what we know about…
Z: You want to do association [makes swapping motion with hands].
M: That's something different, the associative. The commutative is what you're thinking of.
S: Why do we want to do that though?
Z: Because we want to find 3R + 2F.
M: Because our brains work in such a way that…
Z: We always want to simplify.
M: Because we've computed so many times, we've based our ideas on those basic properties of addition and multiplication, and those basic properties are what we rely upon to make our adding and our multiplying easier. So, you have to state that so you know not to depend on those.

Z: I have a different view. According to our stereotype of math, you need to make it simple – your result must be simplified. Like if I have 3R is the identity, then I can use 1 to represent it. Or I add two or subtract two and they cancel, so I can get 0, which means I can simplify.

M: We need to know when to simplify and when not to simplify.

Mary's suggestion that without being told that order matters that one would "start to group similar letters" is based on her understanding of how students assimilate experiences into available ways of operating. She suggests that without the warning, combining like terms is a cognitive necessity, and she justifies her rationale by appealing to her knowledge of how students come to know operations with real numbers. Zowie suggests that since mathematics practitioners are accustomed to a *need* to simplify, the rules must include an explicit exemption from that requirement. In their justification for the retention of the logically superfluous but pedagogically necessary "order matters" rule in a minimal list, the participants demonstrated how their KCS affected their own learning of mathematics.

Discussion

Both the doctoral students and the undergraduates in Larsen's study (2010) questioned the degree to which the symbols they used retained a structure, e.g., whether using a '+' sign would mean that an operation was additive. None of the pairs immediately resolved the issue of the two types of order, and they each created and retained the rule 'order matters.' Larsen (2010) argued that the difficulties that undergraduates had with commutativity and associativity were quite similar to the difficulties found in the literature pertaining to middle-school students and teachers. The current study provides additional evidence of the persistence of this difficulty. Furthermore, the doctoral students' KCS appeared to play a large role in their explanations for their mathematical thinking; they did not sunder their knowledge of how students learn mathematics from their own learning of mathematics.

In the final session the doctoral students expressed that they had been actively looking for ways to connect the abstract algebra they were learning to other content areas (e.g., functions, matrix theory) within the domain of teaching and learning mathematics. Thus, the connectedness of the participants' mathematics knowledge and pedagogical content knowledge may have enabled their construction of the advanced mathematical knowledge they were learning as KMH. One might consider that the mathematics coursework preparation of the participants in this study is not unlike the mathematics preparation of secondary teachers when they first learn about abstract algebra, and, in some cases, nearly identical to the mathematics preparation of in-service elementary or middle school teachers. Therefore, the results suggest that even if teachers do not take an abstract algebra course, they may build KMH from an opportunity to engage in similar mathematical activities that engender reflection on the properties of binary operations.

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