CONCEPTUALIZING VECTORS IN COLLEGE GEOMETRY: A NEW FRAMEWORK FOR ANALYSIS OF STUDENT APPROACHES AND DIFFICULTIES

Oh Hoon Kwon

Department of Mathematics
University of Wisconsin Madison

This article documents a new way of conceptualizing vectors in college geometry. The complexity and subtlety of the construct of vectors highlight the need for a new framework that permits a layered view of the construct of vectors. The framework comprises three layers of progressive refinements: a layer that describes a global distinction between physical vectors and mathematical vectors, a layer that recounts the difference between the representational perspective and the cognitive perspective, and a layer that identifies ontological and epistemological obstacles in terms of transitions towards abstraction. Data was gathered from four empirical studies with ninety-eight total students to find evidence of the three major transition points in the new framework: physical to mathematical coming from the first layer, geometric to symbolic and analytic to synthetic from the second layer, and the prevalence of the analytic approach over the synthetic approach while developing abstraction enlightened by the third layer.

Key words: vector, geometry, representation, vector representation

The complexity and subtlety of the construct of vectors motivate the necessity of the new framework that permits: differentiating abstraction from physical embodiment, intertwining the representational perspective and the cognitive perspective on vectors, and revealing cognitive development on geometric representations. These needs guided us to deliver a new framework for conceptualizing the construct of vectors. By building a new framework and validating it, I explored a complex construct of vectors in mathematics with respect to mathematical abstraction, multiple representations, and cognitive development.

The primary goal of the framework is to give a new way to discuss the complexity of vectors, both conceptual and pedagogical, that students may grapple with in order to understand vectors in geometry effectively.

Complexity of the Construct of Vectors

Vector as a Translation

When representing a geometric translation with a vector in an arrow form, the complexity of vector representations comes out in a cognitive sense. Recent critical studies on students' experiences with vectors focus on physics education (Aguirre and Erickson, 1984; Aguirre, 1988; Hestenes et al., 1992; Heller and Huffman, 1995; Knight, 1995; Savinainen and Scott, 2002; Nguyen and Meltzer, 2003; Flores et al., 2004; Coelho, 2010). These studies focused more on the interrelationship among physical quantities, not on vectors themselves. These physical embodiments of vectors help students understand vectors initially, but soon block the progression to advanced and abstract understanding of vectors.

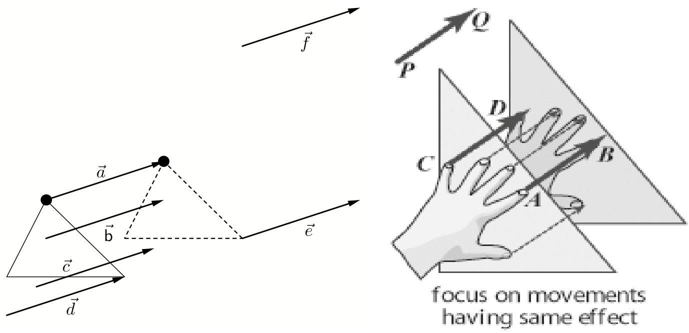


Fig. 1.1 Vector as a Translation Problem

For example, in Figure 1.1 physically all the vectors can show different meanings such as push, pull, moving vertices, and penetrating etc. However, all the vectors represent the same translation of a triangle even though their locations are quite scattered and different. The figure also shows all the vectors are equivalent just as all the vectors represent the same translation, although their equivalence is not clearly explained by the translation action with the given triangle and the same effect those vectors would bring. This difference between physical motion and mathematical motion (Freudenthal, 1983), and the idea of the vector equivalence relation with ‘action-effect’ approach (Watson et al., 2003; Watson, 2004) show the difference between physical embodiment and mathematical abstraction.

For a better observation and description of this complexity and subtlety of vectors, I need a new framework that integrates and balances these intuitive understanding and abstract understanding of vectors in college mathematics.

Vector as a Point, Point as a Vector

The complexity of vectors does not allow students an easier translation/conversion from one representation to the other. A combined view of the representational perspective and the cognitive perspective helps us understand this translation/conversion from one representation to the other.

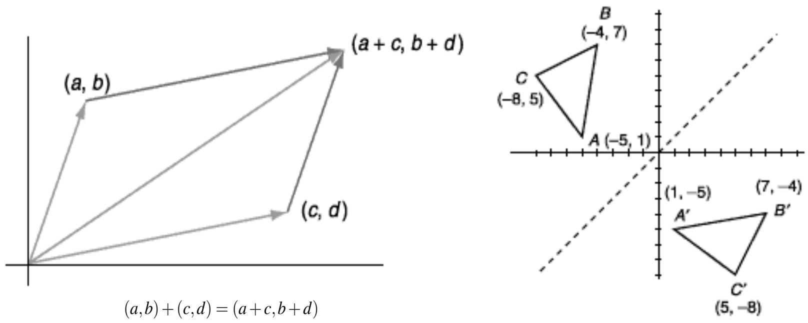


Fig. 1.2 Vector as a Point, Point as a Vector Problem (NCTM 2000)

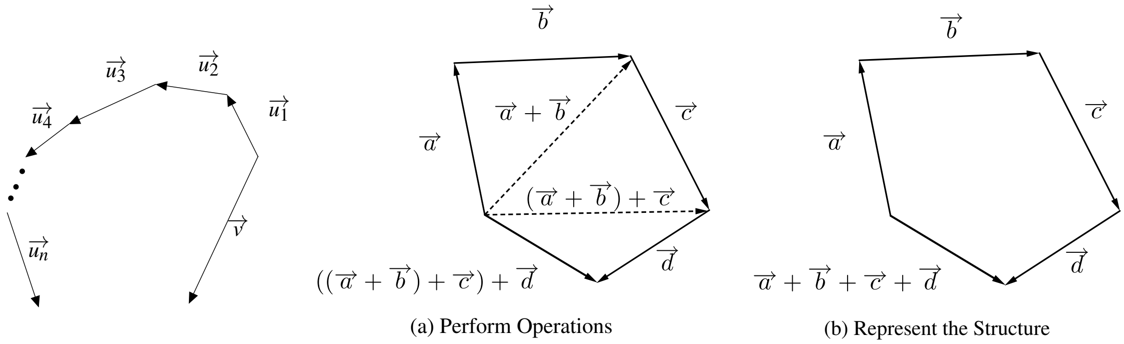
In Figure 1.2, the vector sum in arrow forms is usually accompanied by 2-tuple of numbers. However, it is not clear if this 2-tuple represents the terminal point of the arrow or the arrow itself. When geometry meets linear algebra, this problem of multiple representations gets more complicated. The term ‘vectors’, ‘points (vertices)’, and ‘arrows’ are not easily distinguishable in these two examples. Hillel (2002) criticized that in practice, most instructors tended to shift back and forth between the arrow and point depiction of vectors ‘implicitly’ and ‘unconsciously’ with three modes of description. The study of modes of thinking by Sierpinska (2002) reveals an expansion of the discussion as a problem of how students think about representations with focusing more on students’ thinking and reasoning about epistemology. These classifications of vectors motivated me to consider multiple perspectives on vectors: representational and cognitive.

Research in the mathematics education community posits that students can grasp the meaning of mathematical concepts by experiencing multiple mathematical representations (Janvier, 1987; Kaput, 1987a; Keller and Hirsch, 1998). In this context, several standards documents have advocated K-12 curricula that emphasize mathematical connections among representations (NCTM, 1989, 2000; CCSSM, 2010). They suggest that students use graphical, numerical, and algebraic representations to investigate concepts, problems, and express results. However, these discussions are very focused on the way to talk about functions, not vectors, and on the way to discuss connections between representations as separate entities from representations themselves. Studies on multiple representations of vectors are few and focused on the views from linear algebra (Dorier, 2002; Harel, 1989; Dorier and Sierpinska, 2001). The construct of vectors is more complex than functions, so that graphical, numerical, and algebraic representations are not enough to describe this complexity (Pavlopoulou, 1993 as cited in Artigue et al. 2002).

These observations on translation/conversion bring a need for a unified, inclusive, and multidimensional framework to discuss a combined view of representational and cognitive perspectives on the complexity of the construct of vectors.

Geometrical Vector Sum

What the classical representations cannot provide from the complexity of vectors is the cognitive development of geometric representations. Specifically, translations/conversions in terms of the cognitive perspective are portrayed in considerable detail as cognitive development theories for symbolic representations. Pavlopoulou's research (as cited in Artigue et al., 2002) studied translations/conversions from one representation to the other (Duval, 2006). However, it is very restricted to certain forms of vectors: graphical, table, and symbolic representations (registers). The Action-Process-Object-Schema theory (Asiala et al., 1996) and reification theory (Sfard and Linchevski, 1994) are based on the duality of the mathematical concepts and on the assumption that process conception precedes object conception. Sfard (1991) calls process conception operational outlook and object conception structural view. However, these studies are restricted to discussions of symbolic representations. In terms of geometric representations, Figure 1.3 describes interesting observations about process-object duality (Sfard, 1991; Gray and Tall, 1993, 2001; Forster, 2000).



(a) Perform Operations (b) Represent the Structure

Fig. 1.3 Geometrical Vector Sums

When students calculate a vector sum with the tip to toe triangle method, the process view of a single vector is described as shifting or moving a particle. As a result of the sum, students can put the resulting vector on the appropriate position, and make the sum itself as an object in the structure of a triangle. On the other hand, in parallelogram method, vectors are objects and moving the object with equivalence relation to draw a parallelogram is the process of sum. This example shows that process and object are preceders and succeedors of each other and shows further need for cognitive development as a part of the framework.

The problem of identifying cognitive development in geometric representations of vectors poses the need for a new framework that can show the representational and the cognitive obstacles more clearly in terms of the transitions towards mathematical abstraction. This new framework brings the complexity of vectors to the surface so that one can capture the whole picture of encapsulation or reification both happening in a symbolic way and a geometric way simultaneously in the construct of vectors.

Construction of the Configuration

Those needs that I discussed in the previous chapter grounded this development of the framework that allows a layered view to see the complex construct of vectors. Three layers of progressive refinements are introduced sequentially. They have different scales of focus from aggregate and global dealing with the difference of physical and mathematical vectors (first layer) to individual and local dealing with the representational and the cognitive obstacles (third layer). The final construction of a configuration of vectors as a new framework based on the process of refinement grounded by the needs in the previous chapter is in Figure 2.1 (Kwon, 2011).

In the domain of mathematical vectors, each axis was hypothesized to have two important transitions that can be identified in the configuration. On the epistemological axis, there are (1) one from arithmetic to algebraic, and (2) one from analytic (procedural) to synthetic (structural). On the ontological axis, there are (1) one from geometric to symbolic, and (2) one from concrete to abstract. Among those four transitions, we will only focus on two transitions: analytic to synthetic and geometric to symbolic in this article.

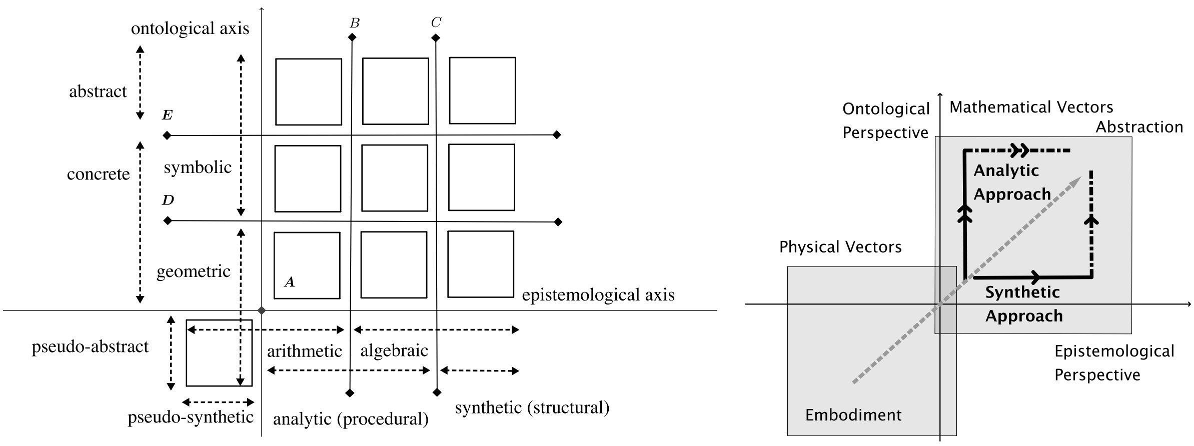


Fig. 2.1 The Configuration of Vectors and Two Approaches towards Abstraction

In this framework, three layers of progressive refinements are comprised. Assuming a difference of mathematical vectors and physical vectors, I first set up two categories: mathematical vectors and physical vectors. Direction of intended movement is the direction of the development towards mathematical abstraction wanted from students. The second layer describes the difference between the representational and the cognitive perspectives on vectors as the difference between the ontological and the epistemological perspectives. The third layer identifies the representational and the cognitive obstacles in terms of transitions towards abstraction. See Figure 2.2.

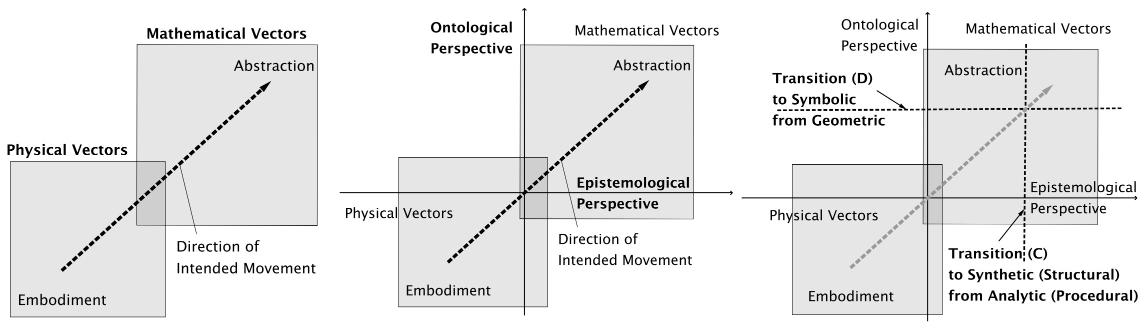


Fig. 2.2 Three Layers of Progressive Refinement

The configuration also shows the analytic approach and the synthetic approach towards mathematical abstraction described by the second and third layers. In terms of the configuration, I can describe the analytic approach as a trend of changing explicit representations along the ontological axis quickly from geometric representations such as arrows, to symbolic representations such as coordinate/column vector forms. Epistemological development is postponed and following symbolically after the ontological change. This analytic approach in the configuration can be illustrated as an upside down 'L' shape route towards abstraction. The synthetic approach is defined as the trend of changing the views/thinking of a geometric representation while maintaining arrow forms, from analytic (procedural) to synthetic (structural) first. Ontological development is postponed until the achievement of the change in epistemological perspectives such as from analytic (procedural) to synthetic (structural). This approach in the configuration marked as a reversed 'L' shape route towards abstraction. See Figure 2.1.

Empirical Studies

I will discuss the method for gathering evidence from student data to see if this layered view of the construct of vectors as a result of progressive refinements is reasonable.

Research Focuses

The focus is on evidence in student data of the following features that are hypothesized by the configuration of the construct of vectors: (1) the three major transition points: (A) physical to mathematical coming from the first layer, (C) analytic to synthetic, and (D) geometric to symbolic coming from the second and the third layers of the framework, (2) process-object duality in geometric representations, and (3) the prevalence of the analytic approach to the synthetic approach while developing mathematical abstraction.

Method and Participants

Four surveys and interviews were carried out to gather evidence on the important features suggested by the configuration. The results are a synthesis of the data gathered from total ninety-eight students who are pre-service secondary and elementary/middle level teachers. Multiple administrations were used to: (1) test appropriate survey questionnaire and interview process, (2) gather deeper knowledge of student background and idea on vectors, and (3) modify surveys and interviews in order to avoid any confusion derived from the questions. All data were collected from students located in the Midwest public university.

After administering each survey, students were selected for interviews based upon their response. The selected students signed up for a one-hour block of time for their interviews on the day and time that was most convenient for them. Interviews were held in a neutral location away from the students' classrooms and were audio-recorded for further analysis. Transcribed interviews were coded and analyzed in order to find evidence in student work. Descriptive statistics were used for the second study and the third study on which a sufficient number of participants were available.

Design and Construction of Surveys and Interviews

For the in-depth discussion of the research focus, I chose the following questions from the surveys and synthesized the results. See Table 3.1. Interviews were conducted with repeating the survey questions and asking further about what the students were thinking. Because questions for examining the prevalence of an approach to the other in the surveys were only asked to students to choose the representations, I asked students to proceed and finish the proof in the interview sessions.

Key Features	Related Layers	Transitions	Questions
Physical vs. Mathematical	I	A	Translation, (Translation of Polygon), (Geometric Translation), Rainy Day, Robot Arm
Epistemological Diff. & Ontological Diff.	II	A, C A, D	Translation, Polygon Rainy Day, Robot Arm
Epistemological Obst. & Ontological Obst.	III	(A), C (A), D	Polygon, Very Long Sum Origin, Robot Arm
Process-Object Duality in Geometric Representation	III	C	Very Long Sum
Prevalence of Analytic Approach	II & III	C, D	Cube, △ Midpoints, Associativity

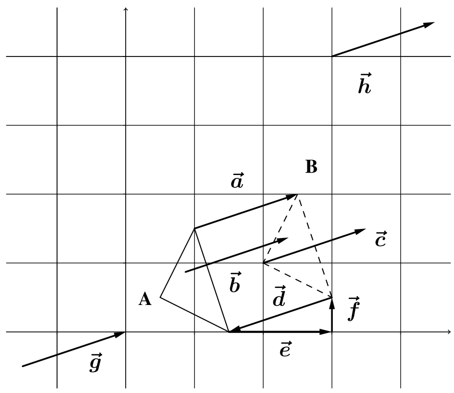
Table 3.1 Questions for Layered View of Configuration

Findings and Discussions

In this chapter, I provide evidence of some focal points (Transition A, C, and the prevalence of the analytic approach) that I hypothesized in Research Focus after careful analyses of the four consecutive empirical studies.

The global difference between mathematical abstraction and physical embodiment is evident in student work (Transition A). Student data showed evidence of the different interpretations between 'the same translations' and 'the equivalent vectors'. We usually assume that the concept of the vector equivalence relation in physics is the same with that in mathematics, because 'directions' and 'magnitudes' of vectors are used to verify equivalent relations in both fields. This means that the equivalent vectors are always representing the same translations and vice versa both in physics and mathematics. However, student work for 'Translation' question (Fig. 4.1) shows the difference of the interpretation between the same translations and the equivalent vectors. The cluster tree diagrams (Fig. 4.2) from hierarchical clustering with Euclidean distance are supposed to show similar categorizations assuming 'the same translations' and 'the equivalent vectors' are the same concept. The first cluster tree was made from the student responses to question (a), and the second cluster tree was made from those to (b). They show two different categorizations in Figure 4.2.

Translation: A translation can be represented by a vector $\overrightarrow{v}$. $T_{\overrightarrow{v}}(P) = P + \overrightarrow{v}$ for any point P.



(a) List all vectors that do **NOT** represent the translation of triangle A to triangle B in the figure.

(b) List all vectors that are equivalent to $\overrightarrow{d}$.

Fig. 4.1 Translation Question

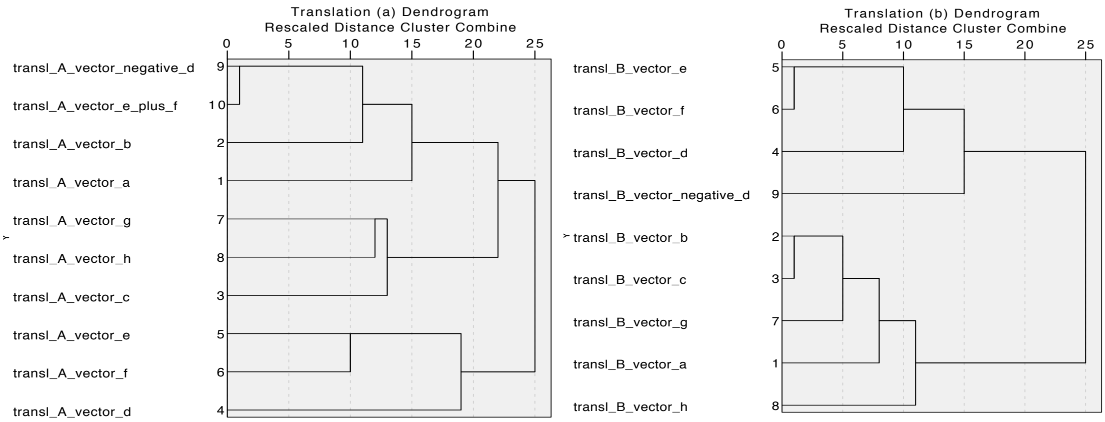


Fig. 4.2 Different Interpretation between Same Translations and Equivalent Vectors

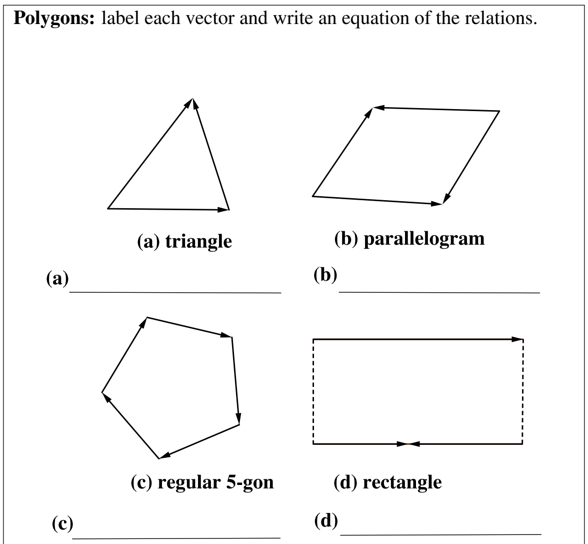
The existence of an epistemological obstacle as Transition (C) from procedural (analytic) to structural (synthetic) is also evident in student work for 'Polygon' and 'A very long sum' questions. This obstacle prevents students to continue calculating the binary vector sum. For 'Polygon' question, students' responses told us that question (a), (b), and (d) based on a triangle, a parallelogram, a rectangle figure were easier for students, but question (c) with a pentagon was harder to solve than other questions (Fig. 4.3). The following interview also shows evidence of Transition (C).

R: "Why did you draw these middle line segments? What are they?"

S: "I think I was trying to do the vector addition and [...] I couldn't really find, based on the method I was trying, couldn't find the way to express the relationship together from a polygon or from a pentagon. [...] Normally I've never seen vectors arranged in that kind of relationship. I've seen them in the triangle, [...] usually in many of these, a parallelogram, a four sided figure, but nothing like this one."

From the interview with the student above, I could see that the student drew a parallelogram to figure out the sum. Thinking the sizes and the directions of arrows compared to thinking the structure that vectors lie on can be regarded as procedural thinking, because the sum was a binary operation and we needed those information for a binary operation. It is evident that there is an obstacle that prevents students using synthetic vectors or structural thinking.

I could also see that students tended to use particular representations more and confine their understanding and using vectors in one approach rather than having flexibility of using both. This tendency was identified in the responses as the prevalence of the analytic approach to the synthetic approach. Because this prevalence is studied and regarded as a trend, and not a specific student's preference, I used the collective data of twenty-nine students rather than concentrating on specific cases. 'Cube', 'Triangle Midpoints', 'Associativity' are questions specially designed to look into the prevalence of the analytic approach to the synthetic approach. The results in Table 4.4 show that the prevalence of the analytic approach to the synthetic approach that I hypothesized is evident in student work.



question	(a)	(b)	(c)	(d)
correct	72.4%	65.5%	**48.3%**	69%
not correct	27.6%	34.5%	**51.7%**	31%
total	100 %	100 %	**100 %**	100 %

Table 4.4: Polygon Result from Survey II

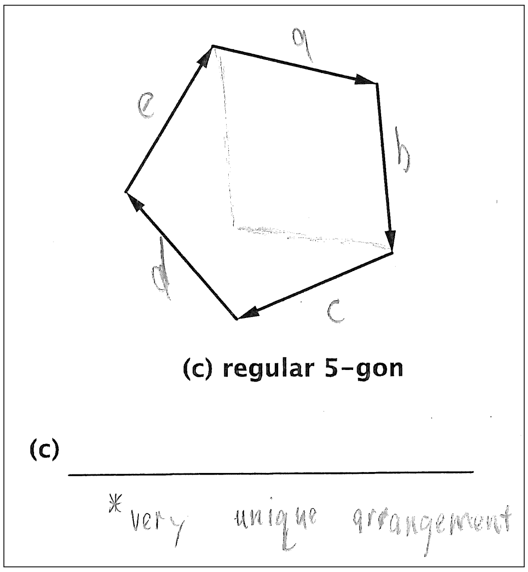


Fig. 4.3 Polygon Results and "Very unique arrangement"

	Analytic Approach	Synthetic Approach	No response
Cube:	**41% (12)**	31% (9)	28% (8)
△ Midpoints:	**55.5% (16)**	34.5% (10)	10% (3)
Associativity:	**72.4% (21)**	24.1% (7)	3.4% (1)

Table. 4.4 Prevalence Results

In summary, I saw that the following are evident in student work when discussing the complexity/subtlety of vectors: (1) the difference between physical and mathematical vectors, (2) the multiple perspectives: ontological and epistemological, and interplay between those two, (3) the prevalence of the analytic approach to the synthetic approach, (4) an epistemological obstacle defined as Transition (C) and an ontological obstacle defined as Transition (D), (5) process-object duality on geometric representations of vectors. What stands out most from these empirical studies is this new framework is very helpful when talking about the complexity and subtlety of the construct of vectors. However, these tentative conclusions with the configuration await further refinement and correction in the light of further research.

While conducting empirical studies, unexpected evidence from the three progressive refinements of the construct of vectors is also shown up as limitations. These limitations include: (1) non-empty intersection between mathematical vectors and physical vectors, (2) unreasonable levels of sophistication reflected in the direction towards abstraction, and (3) problems in repetition of transitions and revered transitions in different contexts. These limitations suggest further refinement and correction of the framework as well as the implications for teaching and learning of vectors.

References

Aguirre, J. and Erickson, G. (1984). Students' conceptions about the vector characteristics of three physics concepts. *Journal of Research in Science Teaching*, 21(5):439–457.

Aguirre, J. M. (1988). Student preconceptions about vector kinematics. *The Physics Teacher*, 26(4):212–216.

Artigue, M., Chartier, G., and Orier, J.-L. (2002). Presentation of other research works. In Dorier, J.-L., editor, *On the Teaching of Linear Algebra*, volume 23 of *Mathematics Education Library*, pages 247–271. Springer Netherlands. 10.1007/0-306-47224-4.

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., Thomas, K., Kaput, J., Schoen- feld, A. H., and Dubinsky, E. (1996). A framework for research and curriculum development in undergraduate mathematics education. *Research in collegiate mathematics education II*, 1:35.

Beckmann, S. (2005). *Mathematics for elementary teachers*. Addison-Wesley.

Brousseau, G., Balacheff, N., Cooper, M., Sutherland, R., and Warfield, V. (1997). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970-1990*. Kluwer aca- demic publishers.

Bryant, P. (1984). Geometry, statistics, probability: Variations on a common theme. *The American Statistician*, 38(1):38–48. ArticleType: research-article / Full publication date: Feb., 1984 / Copyright c 1984 American Statistical Association.

CCSSM (2010). Common core state standards for mathematics. *Retrieved June*, 2:2010. Chevallard, Y. (1988). On didactic transposition theory: Some introductory notes. In *International Symposium on Research and Development in Mathematics, Bratislava, Czechoslavakia*.

Coelho, R. L. (2010). On the concept of force: How understanding its history can improve physics teaching. *Science & Education*, 19(1):91–113.

Dorier, J. (2002). *On the Teaching of Linear Algebra*.

Dorier, J. and Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. *The Teaching and Learning of Mathematics at University Level*, pages 255–273.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. *Advanced mathe- matical thinking*, pages 95–126.

Dubinsky, E. and Tall, D. (1991). Advanced mathematical thinking and the computer. *Advanced mathematical thinking*, pages 231–248.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1):103–131.

Edwards, B. S., Dubinsky, E., and McDonald, M. A. (2005). Advanced mathematical thinking. *Mathematical Thinking and Learning*, 7(1):15.

Flores, S., Kanim, S. E., and Kautz, C. H. (2004). Student use of vectors in introductory mechanics. *American Journal of Physics*, 72(4):460.

Forster, P. (2000). Process and object interpretations of vector magnitude mediated by use of the graphics calculator. *Mathematics Education Research Journal*, 12(3):269–285.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Springer. Goldenberg, E. P. (1995). Multiple representations: A vehicle for understanding understanding. *Software goes to school: Teaching for understanding with new technologies*, page 155–171.

Gray, E. and Tall, D. (1993). Success and failure in mathematics: the flexible meaning of symbols as process and concept. *Mathematics Teaching*, 142:6–10.

Gray, E. and Tall, D. (2001). Relationships between embodied objects and symbolic procepts: an explanatory theory of success and failure in mathematics. In *PME CONFERENCE*, volume 3, pages 3–65. Citeseer.

Gueudet, G. (2004). Geometric thinking in a n-space.Gueudet-Chartier, G. (2004). Should we teach linear algebra through geometry? *Linear algebra and its applications*, 379:491–501.Gueudet-Chartier, G. (2006). Using geometry to teach and learn linear algebra. *Research in Collegiate Mathematics Education*, 6:171–195.

Harel, G. (1989). Learning and teaching linear algebra: Difficulties and an alternative approach to visualizing concepts and processes. *Focus on Learning Problems in Mathematics*.Harel, G. and Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning*, 7(1):27–50.

Heid, M. K., Harel, G., Ferrini-Mundy, J., and Graham, K. (2000). Advanced mathematical think- ing: Implications of various perspectives on advanced mathematical thinking for mathematics education reform. In *Proceedings of the 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, volume 1, pages 33–37.

Heller, P. and Huffman, D. (1995). Interpreting the force concept inventory. *A Reply to Hestenes and Halloun–In: The Physics Teacher*, 33:507–511.

Hestenes, D., Wells, M., and Swackhamer, G. (1992). Force concept inventory. *The physics teacher*, 30(3):141–151.

Hillel, J. (2002). Modes of description and the problem of representation in linear algebra. In *On the Teaching of Linear Algebra*, pages 191–207.

Janvier, C. (1987). *Problems of representation in the teaching and learning of mathematics*. L. Erlbaum Associates.

Janvier, C., Girardon, C., and Morand, J. C. (1993). Mathematical symbols and representations. *High school mathematics*, page 79.

Johnson, S. (1998). What's in a representation, why do we care, and what does it mean? examining evidence from psychology. *Automation in Construction*, 8(1):15–24.

Kaput, J. J. (1987a). Representation systems and mathematics. *Problems of representation in the teaching and learning of mathematics*, pages 19–26.

Kaput, J. J. (1987b). Towards a theory of symbol use in mathematics. *Problems of representation in the teaching and learning of mathematics*, pages 159–195.

Keller, B. A. and Hirsch, C. R. (1998). Student preferences for representations of functions. *Inter- national Journal of Mathematical Education in Science and Technology*, 29(1):1–17.

Knight, R. D. (1995). The vector knowledge of beginning physics students. *Physics Teacher*, 33:74–74.

Kwon, O. H. (2011). Student approaches and difficulties in understanding and use of vectors. In Brown, S., Larsen, S., Marrongelle, K., and Oertman, M., editors, *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education*, volume 1, pages 187 – 202. SIGMMA RUME.

Lakoff, G. and Nunez, R. (2001). *Where Mathematics Comes from: How the Embodied Mind Brings Mathematics into Being*. Basic Books, reprint edition.

Lesh, R., Post, T., and Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. *Problems of representation in the teaching and learning of mathematics*, pages 33–40.

Meissner, H. (2001a). Encapsulation of a process in geometry. *PME CONFERENCE*, 3:3–359. Meissner, H. (2001b). Procepts in Geometry. *European Research in Mathematics Education II*, page 58.Meissner, H., Tall, D., et al. (2006). Proceptual Thinking in Geometry. *RETIREMENT AS PROCESS AND CONCEPT*, page 165.

NCTM (1989). *Curriculum and evaluation standards for school mathematics*. NCTM.

NCTM (2000). Principles and standards for school mathematics. *Reston, Va.: National Council of Teachers of Mathematics*.

Nguyen, N.-L. and Meltzer, D. E. (2003). Initial understanding of vector concepts among students in introductory physics courses. *American Journal of Physics*, 71(6):630–638.

Rider, R. L. (2004). *The Effects of Multi-Representational Methods on Students' Knowledge of Function Concepts in Developmental College Mathematics*. PhD thesis.

Robert, A. (2002). Level of conceptualization and secondary school math education. In *On the Teaching of Linear Algebra*, pages 125–131.

Savinainen, A. and Scott, P. (2002). The force concept inventory. *Physics Education*, 37:45–52. Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational studies in mathematics*, 22(1):1–36. Sfard, A. and Linchevski, L. (1994). The gains and the pitfalls of reification—The case of algebra. *Educational studies in mathematics*, 26(2):191–228.

Sfard, A. and Thompson, P. (1994). Problems of reification: Representations and mathematical objects. In *Proceedings of the XVI Annual Meeting of the International Group for the Psychology of Mathematics Education–North America*, pages 1–32.

Sierpinska, A. (2002). On some aspects of students' thinking in linear algebra. In *On the Teaching of Linear Algebra*, pages 209–246.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. *Handbook of research on mathematics teaching and learning*.

Tall, D. (2004a). Building theories: The three worlds of mathematics. *For the Learning of Mathematics*, 24(1):29–32.

Tall, D. (2004b). Thinking through three worlds of mathematics. In *Proceedings of the 28th Conference of the International*, volume 4, pages 281–288.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2):5–24.

Tall, D., Thomas, M., Davis, G., Gray, E., and Simpson, A. (1999). What is the object of the encapsulation of a process? *The Journal of Mathematical Behavior*, 18(2):223–241.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. *The ideas of algebra, K-12*, 8:19.

Watson, A. (2004). *Effect as a pivot between actions and symbols: the case of vector*. PhD thesis, University of Warwick.

Watson, A., Spyrou, P., and Tall, D. O. (2003). The relationship between physical embodiment and mathematical symbolism: The concept of vector. *The Mediterranean Journal of Mathematics Education*, 1(2):73–97.

Watson, A. and Tall, D. (2002). Embodied action, effect and symbol in mathematical growth. In *PME CONFERENCE*, volume 4, pages 4–369.