STUDENTS' AXIOMATIZING IN A CLASSROOM SETTING

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The purpose of this paper is to examine descriptive axiomatizing as a classroom mathematical activity. More specifically, if given the opportunity, how do students select axioms and how might their intellectual needs influence these decisions? These two case studies of axiomatizing address these questions and elaborate on how students engage in this practice within a classroom setting. The results of this research suggest that while students may at first be resistant to axiomatizing, this mathematical activity also affords them opportunities to create meaning for new mathematical content and for the axiomatic method itself.

Key words: Axiomatizing**,** Intellectual Need, Defining, Realistic Mathematics Education

Introduction

In discussing the genesis of the word, mathematizing, Freudenthal (1990) states that "Mathematizing as a term was very likely preceded and suggested by terms such as axiomatizing, formalizing, schematizing, among which axiomatizing may have been the very first to occur in mathematical contexts" (p. 30). Indeed, axiomatizing is essential to the practicing mathematician, as it serves to create and reorganize knowledge into a fundamental starting point for deductive work in both education and research, but what does it mean for mathematics students to axiomatize?

Theoretical Perspective

While Freudenthal (1973) acknowledges that axiomatizing can be traced back to the classical age of Greece, he cautions that the current usage of axioms is quite different from the function they served in Antiquity: "Axiomatics, as we now use this term, is a modern idea, and ascribing it to the ancient Greeks is, in spite of precursors, an anachronism" (p. 30). Accordingly, axiomatizing is often characterized along ontological lines with "Greek" axiomatizing relying upon selecting self-evident or observable truths, while "modern" axiomatizing is born out of logical convenience and therefore, the axioms one chooses may or may not be self-evident.

In his paper synthesizing much of the previous literature on axiomatizing, De Villiers (1986), identified two fundamentally different types—constructive and descriptive axiomatizing. *Constructive* (a priori) *axiomatizing* occurs when an existing set of axioms is altered through the omission, generalization, substitution or the addition of axioms resulting in content that can then be organized into a new logical structure. Constructive axiomatizing can be illustrated historically by the systematic discoveries and subsequent inventions of non-Euclidean geometries, and its primary function is the creation of new knowledge. On the other hand, "*Descriptive* (a posterior) *axiomatizing* is meant the **selection** of an axiom set from an already existing set of statements" (De Villiers, p. 6)". As a component of systematization, descriptive axiomatizing generally functions to reorganize existing knowledge into a starting point for deductive activity. The selection of axioms may originate with self-evident truths, as in the case of Euclid's geometry, but this is not a requirement of descriptive axiomatizing.

As noted by Freudenthal (1973), many mathematical systems may be too large or too complex for undergraduate students to axiomatize, so a structure such as a group might be more manageable for them. Hence, axiomatizing as a student activity might be readily observable if students are given the opportunity to define the group concept. More specifically, if students are encouraged to select and refine the group axioms, as opposed to being presented with them in some pre-structured form, axiomatizing can then be viewed as a mathematical process that supports their definition of group. This view is consistent with Zandieh and Rasmussen (2010), who stress the importance of including other mathematical activities as part of the act of defining as students engage in "formulating, negotiating, and revising a mathematical definition" (p. 59).

Harel's (1998) Necessity Principle states, "Students are most likely to learn when they see a need for what we intend to teach them, where by 'need', is meant intellectual need, as opposed to social or economic need" (p. 501). The Necessity Principle puts forth a conjecture about how students learn (Speer, Smith & Horvath, 2010) and has been used extensively by Harel as a component of a larger conceptual framework called DNR (Duality, Necessity, and Repeated-Reasoning) (Harel, 2001). As students move "from more computational mathematics courses to upper division, more abstract, mathematics courses such as modern algebra and advanced calculus" (Selden & Selden, 1995, p. 135), one should not be surprised that there is an increased emphasis placed upon structure and axiomatic reasoning. Therefore, Harel's (2011) *intellectual need for computation* (to quantify and calculate) and *the need for structure* (to re-organize knowledge into a logical system) serve as useful constructs for investigating student axiomatizing.

Background

The setting for examining student axiomatizing consisted of two different mathematics classes that used the same inquiry-oriented curriculum for reinventing the concept of group (Larsen, 2009). An overview of the group theory unit in the TAAFU (Teaching Abstract Algebra for Understanding) curriculum is provided here:

> The reinvention of the group concept begins with an investigation of the symmetries of an equilateral triangle (Larsen, 2009; Larsen & Zandieh, 2007). Students develop symbols for the six symmetries and develop a calculus for computing combinations of symmetries. The rules the students use to compute combinations include the group axioms along with relations specific to this group. Based primarily on their work within the symmetry context, the students construct a formal definition of a group (Johnson & Larsen, 2012, p. 3).

Two community college mathematics instructors (Mr. A and Mr. B) used these group theory materials in mathematics bridge courses that were primarily designed to expose students to university mathematics in an active and supportive environment. The elective nature of these courses not only gave the instructors ample time and flexibility to engage in the TAAFU tasks, but to also pursue discussions and activities that originated from the students.

The video data for my research consisted of both whole-class and small-group episodes, taken from the two bridge classes. The primary domain of inquiry occurred during the implementation of a subset of activities in the TAAFU group theory curriculum that are described here. Following their symbolizing activity that consisted of negotiating a class-wide convention for representing the symmetries of an equilateral triangle, the students were then asked to compute all of the combinations of any two symmetries. During this process, they were encouraged to keep track of any "rules" or "shortcuts" they may have used in these computations. After sharing some of these rules in small groups, a class list was then recorded by the teachers, with little regard for economy. For example, the property that three consecutive 120-degree rotations or two consecutive flips essentially did nothing might be represented by the rules $R^3 = I$ or $F^2 = I$ respectively. After a comprehensive list of rules was created, the teachers then asked if all of the rules were necessary or could some of them be deduced from other rules, thus making them redundant. It was during this stage that students put forth different arguments about which rules to keep out of necessity and in the case of equivalent rules, which version to keep, as they tried to create a minimal list. Once the minimal list was agreed upon as a class, the students were then asked to recompute their combinations using only the minimal list of rules and then each student was assigned a particular calculation to prove, again using only the minimized list. Following this exercise, axiomatizing continued to occur sporadically as students progressed toward the formal definition of group, through the refinement of this set of class rules.

Methods

Noting the similarities to De Villier's (1986) notion of descriptive axiomatizing, I conducted a retrospective analysis (Cobb & Whitenack, 1996; Stylianides, 2005) of the students' activity in Mr. A's community college bridge course using "axiomatizing" as the lens of examination. At the conclusion of the first pass through the data, I found the students' activity to be consistent with de Villiers's (1986) notion of descriptive (a posterior) axiomatizing in the Greek sense. In this case, the act of selecting rules was at first done individually, then discussed in small groups, and later collectively negotiated to comprise a more economical list that was ultimately used as a launching point for deductive work. To further analyze the students' progression from informal to formal activity, I made a second pass through this data using Zandieh and Rasmussen's (2010) DMA (Defining as a Mathematical Activity) Framework. This construct provides a lens for examining different levels of activity in the process of defining, namely: situational, referential, general, and formal (Gravemeijer, 1999). During this stage of concept development, I concluded that the students were primarily engaged in referential activity that transitioned to general activity, as they organized a system of relations for the symmetries of an equilateral triangle that later extended to symmetries of other *n*-gons and then eventually to relations for arbitrary sets.

Following my two-stage preliminary analysis, I concluded that de Villier's construct of descriptive defining and Zandieh and Rasmussen's DMA framework globally *described* what the students were doing in the TAAFU classroom episodes, but there were still elements of the axiomatizing sequence that warranted further analysis. For instance, I was curious why some of the students in the bridge course seemed resistant to axiomatizing and why they chose certain rules instead of others. Therefore, I extended my data set to include Mr. B's class and conducted a cross-case analysis of axiomatizing that consisted of two phases: an explanatory pattern-matching analysis followed by a cross-case synthesis (Yin, 2009).

The explanatory pattern-matching analysis served as a starting point for identifying the various phases of the axiomatizing sequence and culminated with explanatory descriptions of the students' activity in both cases. Although other mathematical activities were also occurring during this timeframe, I identified particular milestones (both in terms of class time and in real time) for comparative axiomatic development. As both classes were taught using the same curriculum, patterns such as negotiating notational conventions for the rules and selecting which rules to keep in the minimal set were similar and occurred in the same chronological order. However, given that each list of rules was student generated, the order of the rules and whether certain rules were implicit or explicit constituted notable differences. The cross-case synthesis focused on examining students' intellectual needs as they pertained to axiomatizing. Therefore, I made another pass through the data looking for direct evidence that might explain why students selected certain rules to keep as part of the minimal set and why others were let go.

Results

Following the analysis of the data, several themes emerged, but only two are reported here. First, there was evidence in both case studies of students' initial resistance to axiomatizing. Consistent with Larsen's (2004; 2009), teaching experiments, some of TAAFU students did not see a need for axiomatizing the associative property. Larsen (2009) explained this resistance by noting that in the initial stages of the curriculum, it was not uncommon that students would view algebraic expressions as a sequence of actions, rather than as a binary operation acting on two elements. Although the closure property was identified very early in the curriculum, both groups of students also struggled with axiomatizing a rule for it as well. For instance, when the students used the technique of "multiplying both sides by symmetry" in order to solve a linear equation, both teachers asked which axiom justified this technique. One of Mr. B's students responded, "Isn't that just the multiplicative property of equality?" Even when Mr. B reminded them that the operation was not multiplication, some students suggested that it was obvious from their operation table or that this rule could be deduced using their existing set of axioms.

I also found that some students were resistant to axiomatizing when presented with the minimization task. For the majority of these students, economy was inhibited by what I inferred as an intellectual need for computation. For example, even after it was shown that one rule could be derived from the other, some of the students insisted on keeping two versions of the dihedral relation (i.e. $RF = FR^2$ and $RFR = F$) in their minimal list because it made certain calculations "faster". For a few students in the transition courses, there was even a global resistance to economy as illustrated in the following excerpt from a student in Mr. A's class:

Chris: Ok, so um… I just ran into this whole logical error for this whole situation…imagine you are out shopping for a calculator and you get to the store and there's one calculator that costs like five bucks. It can add, it can subtract, it can multiply, it can divide and that's all. Then there's a graphing calculator for like sixty dollars that can do all this stuff really easily. You're gonna go, I mean if a you are in an advanced math class, you are going to go for the graphing calculator, even though it costs more—even though it is a greater initial expense, uh, because it has all of these pre-programmed into it. Basically what we are doing at this point, um and it just seems like a logical fallacy, is we're reducing the capabilities of our calculator to as few as possible, which just doesn't seem all that efficient.

While there was some initial resistance to axiomatizing, this reluctance seemed to diminish as the students progressed from referential to more general activity. For instance, once Chris had established a need for minimizing the list, he suggested keeping $RFR = F$ as the dihedral rule as opposed to $RF{=}FR^2$, because it applied not only to the triangle, but also to any regular polygon. Students also suggested replacing *F*'s and *R*'s with other letters such as *A*, *B*, and *C*, so they could refer to symmetries of other figures in addition to the symmetries of an equilateral triangle.

A second theme that emerged from the data suggested that descriptive axiomatizing not only provided the students opportunities to structure existing content, but it also fostered a rich context for discussing new student-initiated mathematical ideas. For example, in Mr. A's class, a student put forth a rule that was equivalent to $R^{3n} = F^{2n} = I$, which naturally led into a conversation about modular arithmetic. While this rule was not accepted as a class rule because it was not immediately useful in the task of combining any two symmetries, the student who suggested it noted its future computational value and offered ways to modify the rule so that it would apply to other regular *n*-gons. In Mr. B's class, students chose to axiomatize the law of exponents, which precipitated a discussion about rules that were essential for describing their symmetry relations and those that were either consequences of their notational choice (as in the case of the exponent rule) or more general relations, such as the transitive property of equality.

Structural recommendations also emerged as the students revised their list of rules. For example, one student suggested reordering the list of rules, so that generic symmetries and the identity symmetry were defined into existence before they were used in a subsequent rule. The act of axiomatizing itself was even considered in both of the bridge courses when some students commented on their desire to keep a rule, but somehow relegate it as less important than the others. When a student in Mr. B.'s class asked if this distinction was the difference between a theorem and an axiom, it resulted into a lively whole-class discussion that introduced these terms as well as lemma and corollary to many students who were unfamiliar with them.

Discussion

In proof-based courses, much attention is paid to the deductive stage of the axiomatic method and yet, a common theme that is recurrent in the literature is the disconnection between how students view proof when compared to the mathematical community (Tall, 1989; De Villers, 1990; Almeida, 2000). In light of this study, an analogous conjecture might be made regarding students' views of axioms, which are utilized both implicitly and explicitly in proofs. For instance, the students in this study not only initially struggled with creating rules, but also in knowing when they were using (or not using) a rule in a calculation or a proof. These students also expressed difficulty with discerning objects that are typically viewed by the mathematics community as axioms from properties of an equivalence relation or a relation that was consequence of their notation choice, such as the law of exponents.

On the other hand, this study also suggests that if students are asked to create and refine rules, which describe relations that originate from their own mathematical activity, they may have the opportunity not only to formalize those relations, but also to consider axiomatic qualities as they reorganize their existing knowledge into a logical structure. While such activities may not be considered axiomatizing in the "modern" mathematical sense, the students in this study engaged in rule making that was consistent with "Greek" descriptive axiomatizing. In contrast to being given a pre-structured form of Euclid's *Elements*, which traditionally serves as a starting point for deductive work in Euclidean geometry, these students had opportunities to realize an intellectual need for axiomatizing a mathematical system prior to utilizing those axioms in deductive work.

Conclusion

These case studies suggest that although it may be met with initial resistance, descriptive axiomatizing can be a fruitful activity for students. In a traditional mathematics classroom, students are rarely given opportunities to axiomatize. However, as mathematics students progress through an undergraduate program, the increasing importance of axioms and reasoning from them cannot be ignored. Therefore, if we expect students to understand and reason from axioms, it might be worthwhile to foster an intellectual need for them by giving them opportunities to create and refine axioms.

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