Students Reconciling Notions of One-to-One Across Two Contexts

Michelle Zandieh | Jessica Ellis | Chris Rasmussen
Arizona State University | San Diego State University | San Diego State University

As part of a larger study of student understanding of concepts in linear algebra, we interviewed 10 university linear algebra students as to their conceptions of functions from high school algebra and linear transformation from their study of linear algebra. Analysis of these data led to a classification of student responses into properties, computations and a series of five interrelated clusters of metaphorical expressions. In this paper, we use this classification to analyze students' written and verbal responses to questions regarding one-to-one in the context of function from high school algebra and in the context of linear transformation from their study of linear algebra. We found that students' ability to construe sameness across the two contexts is related to their reliance on properties versus metaphors. We conjecture that this phenomenon is likely to occur for other mathematical constructs as well.

Keywords: Concept image, function, linear algebra, linear transformation, metaphor

The research reported in this paper began as part of a larger study into the teaching and learning of linear algebra. As we examined student understanding of linear transformations we wondered how student understanding of functions from their study of precalculus and calculus might influence their understanding of linear transformations and vice versa. In previous work we created a framework for analyzing student understanding that incorporates five clusters of metaphorical expressions as well as properties and computations that students spoke about when discussing function or linear transformation. In this paper we apply this framework to the setting of students reconciling their understandings of one-to-one in the context of function with their understandings of one-to-one in the context of linear algebra. Ideally we would like students to be able to recognize a similar structure for one-to-one in each context, and thereby to strengthen their overall understanding of the notion of one-to-one. This proposal provides four vignettes that we found illustrative of the way students reasoned about one-to-one within and across the two contexts. More broadly we find the case of one-to-one as prototypical of the struggles students have in reasoning seeing similarities across contexts.

Literature and theoretical background

The nature of students' conceptions of function has a long history in the mathematics education research literature. This work includes Monk's (1992) pointwise versus across-time distinction, the APOS (action, process, object, scheme) view of function (e.g., Breidenbach, Dubinsky, Hawkes, & Nichols,1992; Dubinsky & McDonald, 2001), and Sfard's (1991, 1992) structural and operational conceptions of function. A comparison of these views may be found within Zandieh (2000). More recent work has focused on descriptions of function as covariational reasoning (e.g., Thompson, 1995; Carlson, Jacobs, Coe, Larsen & Hsu, 2002). A recent summary with a focus towards covariational reasoning is found in Oehrtman, Carlson, and Thompson (2008).

The work in linear algebra has tended to focus more on student difficulties (e.g., Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000; Harel, 1989; Hillel, 2000; Sierpinska, 2000). There have been a few studies on student understanding of linear transformation (Dreyfus, Hillel, & Sierpinska, 1998; Portnoy, Grundmeier, & Graham, 2006). However, we could not find studies that relate student understanding of function and linear transformation. In addition we did not yet find a study focused on the notion of one-to-one in either context.

In addition to work specifically on student conceptions of functions or linear transformation, we draw on the notion of concept image and the work done showing that students often have many aspects of their concept image that are not immediately compatible with their stated concept definition (Vinner & Dreyfus, 1989; Tall and Vinner, 1981). Our work is related but focuses more on how a student's concept image of a mathematical construct (in our case one-to-one) may be more or less compatible with their concept image of the same construct in another setting.

In addition to work that uses concept image as its framing, we find useful studies that (whether they refer to it by the term concept image or not) detail student concept images of mathematical constructs using the construct of a *conceptual metaphor* (e.g., Lakoff & Núñez, 2000; Oehrtman, 2009; Zandieh & Knapp, 2006). This follows from the earlier work in cognitive linguistics of Max Black (1977), Lakoff and Johnson (1980) and Lakoff (1987). Following from this work, our assessment is that a person's concept image of a particular mathematical idea will likely contain a number of metaphors as well as other structures.

We rely on metaphorical expressions to indicate when a conceptual metaphor is being employed. Lakoff and Johnson (1980) explain that, "Since metaphorical expressions in our language are tied to metaphorical concepts … we can use metaphorical linguistic expressions to study the nature of metaphorical concepts and to gain an understanding of the metaphorical nature of our activities (p. 456)." A metaphorical expression is an expression that uses metaphorical language, such as describing love as a journey: "they are in the fast lane to marriage"; "our relationship has come to a cross roads". Our framework includes clusters of metaphorical expressions that allow us to highlight the connections or discrepancies between student conceptions in the context of function and the context of linear transformations.

Methods

The data for this report comes from semi-structured interviews with 10 students who were just completing an undergraduate linear algebra course. The interviews were videotaped and transcribed and student written work was collected. In addition to the main interview questions like those listed below, students were often asked follow up questions to gain more insight into their thinking. The focus of the interview was to obtain information about students' concept image of function and their concept image of linear transformation and to see in what ways students saw these as the same or different. To this end we not only asked the students how they thought of a function or linear transformation, but also questions about characteristics that would be relevant to both functions and linear transformations such as one-to-one, onto, and invertibility. For the purposes of this paper, we draw on student responses to questions regarding the concept of one-to-one:

1. In the context of high school algebra, give an example of a function that is 1-1 and one that is not 1-1. Explain.
2. In the context of linear algebra, give an example of a linear transformation that is 1-1 and one that is not 1-1. Explain.
3. Please indicate, on a scale from 1-5, to what extent you agree with the following statement: "1-1 means the same thing in the context of functions and the context of linear transformations."

In order to identify the various ways students reasoned about function, linear transformation, and one-to-one in both contexts, we applied the theoretical framework developed by Zandieh, Ellis, and Rasmussen (2012). This framework details three main components of students' concept images of function and linear transformation: by drawing on *properties,* by drawing on *computations,* or by drawing on *metaphors*. The property ($P_{\text{property used}}$) category refers to student statements that do not delve into the inner workings of the function or transformation. Students also frequently drew upon computational language while reasoning through the interview tasks. We differentiated between computational language that described how a function or linear transformation behaves (labeled as C1) and side computations done involving the function or transformation (labeled as C2).

The metaphorical component consists of five related metaphors that share the common structure of a beginning entity, an ending entity, and a description about how these two are connected, as shown in Table 1. The first metaphor is the *input/ output* metaphor (IO), and involves an input, which goes into something, and an output, which comes out. The second metaphor is *traveling* (Tr)*,* and involves a beginning location being sent or moving to an ending location. The third metaphor is *morphing* (Morph) and involves a beginning state of an entity that changes or is morphed into an ending state of the same entity. The fourth metaphor, *mapping* (Map), most closely resembles the formal Dirichlet-Bourbaki definition of function, and involves a beginning entity, an ending entity, and a relationship or correspondence between the two. The fifth and final metaphor is the *machine* metaphor (Mach), and includes a beginning entity or state, an ending entity or state, and a reference to a tool, machine or device *that causes* the entity to change from the beginning entity/state into the ending entity/state. Because these five metaphors share a common three-part structure, students often layer metaphors on top of one another. In the body of this paper, we will provide multiple examples of this layering of the metaphor clusters.

Table 1: Structure of the metaphor clusters.

Cluster	**Entity 1**	**Middle**	**Entity 2**
Input/Output (IO)	Input(s)	Entity 1 goes/is put into something and Entity 2 comes/is gotten out.	Output(s)
Traveling (Tr)	Beginning Location(s)	Entity 1 is in a location and moves into a (new) location where it is called Entity 2.	Ending Location(s)
Morphing (Mor)	Beginning State of the Entity(ies)	Entity 1 changes into Entity 2.	Ending State of the Entity(ies)
Mapping (Map)	First Entity	Entity 1 and Entity 2 are connected or described as being connected by a mapping (a description of which First entities are connected to which Second entities).	Second Entity

Machine (Mach)	Entity(ies) to be processed	Machine, tool, device acts on Entity 1 to get Entity 2.	Entity after being processed

In this study we are interested in to what degree students are able to reconcile their understandings of one-to-one in the contexts of function and linear transformation. In question 3 above we asked students directly to what extent they saw their understandings as compatible, and through follow up questions we asked them to show us how their understandings were compatible. The ten student interviews included students who clearly showed how the notion of one-to-one is compatible across the two contexts, and students whose notions of one-to-one were not as compatible across contexts. Below we present four vignettes that illustrate the primary variations in student thinking within our group of ten students. Table 2 provides summary information for each of the 10 students including the metaphors and properties that they referred to when answering the interview questions about one-to-one in the context of function (Column 2) and linear transformations (Column 3), their answer to question 3 (Column 4), and whether or not the student reconciled one-to-one across the two contexts (Column 5).

Table 2. Students’ expressed understandings of one-to-one in the contexts of function and linear transformation.

Name	Function	Linear Transformation	Question 3 (scale 1-5)	Reconciled
Donna	IO P_{hlt} $P_{\text{shape of graph}}$ P_{map} P_{vlt} $P_{\text{no holes}}$	IO $P_{\text{different directions}}$ $P_{\text{infinite solutions}}$ P_{map} P_{line} $P_{\text{infinite inputs}}$ P_{ld} P_{li}	3	No
Nila	Map $P_{\text{monotonic}}$ $P_{\text{shape of graph}}$ P_{vlt}	IO Map Tr P_{ld} P_{line} P_{li} P_{map}	1	No
Jerry	IO Map $P_{\text{shape of graph}}$	P_{ld} P_{li}	3	No
Josh	P_{hlt} $P_{\text{shape of graph}}$	P_{li} P_{ld} $P_{\text{invertible}}$ P_{line} P_{diag} P_{span}	3	No
Adam	Map $P_{\text{shape of graph}}$ P_{hlt}	IO Morph Tr C_1 P_{ld} P_{line} P_{li} P_{span}	5	No
Gabe	IO Map	Map Tr P_{square} P_{ld} P_{li} $P_{\text{invertible}}$	5	No
Lawson	IO Map $P_{\text{shape of graph}}$	Map Morph $P_{\text{infinite solutions}}$ P_{ld} P_{map}	5	No
Nigel	Map P_{hlt} $P_{\text{shape of graph}}$	Map Morph Tr	4	Yes
Randall	IO Map	IO Map	5	Yes
Brad	IO Map	IO Map Tr	5	Yes

Note. Students whose names are in italics represent their category in the vignettes below.

Vignettes

We found that students in our study fell into four categories with regard to their belief about and ability to reconcile the similarity of one-to-one across the two contexts. The first group gave low (1 or 3) scores to question 3 and did not reconcile; the second group gave scores of 5 to question 3 and did not reconcile; the third group gave a score of 4 and did reconcile; and the last group gave a score of 5 and reconciled. In the following section we provide a vignette from one student for each of these four categories. The first two vignettes show different ways that students struggled to show that their understandings of one-to-one were compatible across the two contexts, and did not reconcile their understandings. The last two vignettes show different ways that students were able to reconcile their understandings of one-to-one.

Vignette 1. The first vignette tells the story of a student whose understandings of one-to-one relied heavily on properties, which prevented her from seeing her descriptions of one-to-one as compatible. Donna's initial description of one-to-one in the context of function mentioned a mapping understanding, but she relied on the horizontal line test to determine when a function is or is not one-to-one. In the context of linear transformation she did not mention a mapping understanding and instead relied exclusively on linear (in)dependence. When asked if she was thinking of one-to-one the same way in both descriptions, Donna replied that she had not:

Donna: No, I don't think so, because I was thinking in terms of just simple, what I learned in high school, how the one-to-one function is something that has exactly one input for one output. And then in linear algebra, I was thinking in terms of linear dependency and independency and what we had learned prior in the class.

Donna's reliance on properties about one-to-one functions and one-to-one linear transformations prevented her from reconciling these notions.

Vignette 2. The second vignette tells the story of a student who strongly believed that one-to-one is that same in both contexts, but through his descriptions of one-to-one in each context did not completely recognize the underlying structure of one-to-one and eventually became uncertain about the strong similarity of one-to-one in the two contexts. Adam's initial description of one-to-one used properties and mapping language for function, and properties, computational, and input/output language for linear transformation. Specifically, he referenced the shape of a parabola for the reason x^2 is not one-to-one, and the linear independence of the column vectors of the 2 by 2 identity matrix as the reason why it is one-to-one. He strongly agreed that one-to-one meant the same thing in both contexts, and explained that he strongly agreed because "it just feels like the same thing, like if you put 1 in, it only comes out as only 1 possibility."

The interviewer then pointed out that the way he described one-to-one in both contexts "looked kind of different" and asked him to elaborate how he thought about them as the same. Adam then described how he saw linear (in)dependence related to one-to-one, drawing on input/output and traveling language. The interviewer asked him to elaborate examples of function and transformation that are not one-to-one, which he did by referencing properties for both, the mapping metaphor for functions, and computational language for linear transformation. When the interviewer pushed Adam to highlight how these two explanations were compatible, he realized that they were not as compatible as he had originally thought, concluding that his understandings of one-to-one the two contexts are "a little different."

Vignette 3. Nigel is a case of a student who did not initially see a connection between his ways of thinking about one-to-one in terms of function and linear transformation, but then realized a strong connection as he discussed his ideas with the interviewer. When asked question three, Nigel circled 4 and said, "I mean, I would agree, but at the same time, I don't really see a solid connection that I can explain myself."

His initial explanation of one-to-one for function used a property, but morphing ("transformed") and travelling language ("to that spot") for linear transformation:

> Nigel: I've learned of one-to-one means the horizontal line test. … But for linear transformations, I see it as, here's this vector, if it gets transformed by a one-to-one transformation, it's going to get plotted to its own specific new vector, and no other vector will be transformed to that spot.

When asked to reconcile his understandings in the two context, the following exchange occurred.

> Interviewer: So those 2 things sound pretty different on the surface at least, can you say more about the way in which it might be similar for the 2?
>
> Nigel: one-to-one is just for every y value, there's a unique x. … Maybe there's multiple y values for this x, ... so y = sin(x), so it looks something like that. For this, if you have a horizontal line, you just go down, you'll see across here there's … the same y values but for different x values. So, that's not one-to-one.

Nigel used mapping language to describe how the horizontal line test works. His language was more compatible with his description of one-to-one in the context of function, but he used different metaphors in the two contexts. In this interview Nigel never completely reconciled his notions in terms of using identical language but he did find some sense of understanding a compatibility that he had not recognized previously in that he said, "I never really explained it like that before!"

Vignette 4. The fourth vignette illustrates the case of students who can easily reconcile their understandings of one-to-one in the contexts of functions and linear transformations. Each of the two students in this category used similar language when describing examples in each context. For example, Brad discussed a one-to-one function by saying, "This is one output for every input." For a one-to-one transformation he said, "For every output there is one input to get there." In each context Brad used language from the mapping cluster, i.e., for every ___ there is one ___. He also uses input/output language in each context. A subtle difference was his inclusion of travelling language for linear transformation, with the phrase, "to get there." Both Brad and Randall were able to give examples of functions that are not one-to-one in each context and to show that the function or linear transformation is not one-to-one by finding two input values that map to the same output value.

Conclusion

The above four vignettes highlight various ways students understand one-to-one in the contexts of function and linear transformation, and how they explain the compatibilities of these understandings. We see that for the students who rely primarily on properties about one-to-one in the two contexts, such as a function is one-to-one if it passes the horizontal line test, it was difficult for them to identify the consistencies between one-to-one across the contexts. In contrast to this, students who drew on metaphorical language of one-to-one in the two contexts, such as mapping, morphing, or traveling language, the compatibility of the two understandings was more clear.

Each of the metaphorical clusters allow for the description of one-to-one in terms of the relationship between what we refer to in Table 1 as "Entity 1" and "Entity 2". The layering of the metaphorical clusters within one context and the compatibility of the metaphorical language across contexts allows for the recognition of similarities. In contrast, when a student thinks in terms of a property such as the horizontal line test or the linear independence of column vectors in a matrix, then the connections across contexts are more difficult to make. The horizontal line test is specific to the context of a function from the real numbers to the real numbers, and the column vector test is specific to a linear transformation that is defined in terms of matrix multiplication. Either of these properties could be unpacked in terms of one or more of the metaphor clusters (including in terms of the formal definition of one-to-one which fits in the mapping cluster). However, the tests in their most efficient, easy-to-apply format have been condensed or simplified in a way that hides the structural relationships that would allow students to compare across contexts.

This issue speaks to a broader goal of mathematics education: for students to be able to understand a construct, such as one-to-one, across a number of different contexts. One step to helping students develop these broader understandings is by identifying how they understand the construct within various contexts. Our theoretical framework provides such a tool. With this tool we were able to highlight similarities and differences across students understandings of one-to-one in each context. By making these comparisons, it became clear that a reliance on certain properties made developing a more context-free understanding of one-to-one difficult. This phenomenon is likely to occur for other mathematical constructs as well. We see this research as an illustrative example toward exploring this larger phenomenon.

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