Students' Knowledge Resources About the Temporal Order of Delta and Epsilon

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The formal definition of a limit, or the epsilon delta definition is a critical topic in calculus for mathematics majors' development and the first chance for students to engage with formal mathematics. Research has documented that the formal definition is a roadblock for most students but has de-emphasized the productive role of their prior knowledge and sense making processes. This study investigates the range of knowledge resources included in calculus students' prior knowledge about the relationship between delta and epsilon within the definition. diSessa's Knowledge in Pieces provides a framework to explore in detail the structure of students' prior knowledge and their role in learning the topic.

Keywords: limit, formal definition, students' prior knowledge, fine-grained analysis

The formal definition of a limit of a function at a point, as given below, also known as the epsilon-delta definition, is an essential topic in mathematics majors' development that is introduced in calculus. We say that the limit of $f(x)$ as x approaches a is L, and write. $\lim_{x\to a} f(x) = L$ if and only if, for every number ε greater than zero, there exists a number δ greater than zero such that for all numbers x where $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. The formal definition provides the technical details for how a limit works and introduces students to the rigor of calculus. Yet research shows that thoughtful efforts at instruction at most leaves students – including intending and continuing mathematics majors – confused or with a procedural understanding about the formal definition (Cottrill et al., 1996; Oehrtman, 2008; Tall & Vinner, 1981).

Although studies have sufficiently documented that the formal definition is a roadblock for most students, little is known about how students actually attempt to make sense of the topic, or about the details of their difficulties. Most studies have not prioritized students' sense making processes and the productive role of their prior knowledge (Davis & Vinner, 1986; Przenioslo, 2004; Williams, 2001). This may explain why they reported minimal success with their instructional approaches (Davis & Vinner, 1986; Tall & Vinner, 1981). Thus, understanding the difficulty in the teaching and learning of the formal definition warrants a closer look – with a focus on student cognition and with attention to students' prior knowledge. It also calls for a theoretical and analytical framework that focuses on understanding the nature and role of students' intuitive knowledge in the process of learning.

A small subset of the studies have begun exploring more specifically student understanding of the formal definition (Boester, 2008; Knapp and Oehrtman, 2005; Roh, 2009; Swinyard, 2011). They suggest that students' understanding of a crucial relationship between two quantities, epsilon and delta within the formal definition warrants further investigation. Davis and Vinner (1986) call it the *temporal order* between epsilon and delta, that is epsilon first, then delta (p. 295) and found that students often neglect its important role. Swinyard (2011) found that the relationship between the two quantities is one of the most challenging aspects of the formal definition for students. Knapp and Oehrtman (2005) and Roh (2009) document this difficulty for advanced calculus students. This difficulty is also prevalent among the majority of calculus students who struggled with the formal definition in Boester (2008). While studies have shown the existence and prevalence of this difficulty, little is known about why this relationship is difficult for students.

This report is a preliminary analysis of a pilot dissertation data. The dissertation explores how students make sense of the formal definition of a limit in relation to their intuitive knowledge. Specifically, it investigates the micro changes in student understanding of the temporal order of epsilon and delta within the formal definition of a limit. Through a fine-grained analysis of student interviews, this preliminary report focuses on one of the questions that will be explored in the dissertation. What claims do students make about the relationship between delta and epsilon, and what is the range and nature of the resources they use to make these claims?

Theoretical Framework

The Knowledge in Pieces (KiP) theoretical framework (Campbell, 2011; diSessa, 1993; Smith et al., 1993) argues that knowledge can be modeled as a system of diverse elements and complex connections. From this perspective uncovering the fine-grained structure of student knowledge is a major focus of investigation, and simply characterizing student knowledge as misconceptions is viewed as an uninformative endeavor (Smith et al, 1993). Knowledge elements are context-specific; the problem is often inappropriate generalization to another context (Smith et al, 1993). For example, "multiplication always makes a number bigger" is not a misconception that just needs to be removed from students' way of thinking. Although this assertion would be incorrect in the context of multiplying numbers less than 1, when applied in the context of multiplying numbers greater than 1, it would be correct. Paying attention to contexts, KiP considers this kind of intuitive knowledge a potentially productive resource in learning (Smith et al., 1993). This means that instead of focusing on efforts to replace misconceptions, KiP focuses on characterizing the knowledge elements and the mechanisms by which they are incorporated into, refined and/or elaborated to become a new conception (Smith et al., 1993). Similarly, we view students' prior knowledge as potentially productive resources for learning. We also assume that student knowledge is comprised of diverse knowledge elements and organized in complex ways, and thus learning is seen as the process of reorganization and elaboration of students' prior knowledge.

Methods

The data for this report comes from the pilot data for one of the author's dissertation. We interviewed seven calculus students using a protocol developed for the dissertation. Each of these students has received some form of instruction on the formal definition. So we anticipate some knowledge about the definition to be a part of their prior knowledge. The protocol was designed to elicit student understanding of the formal definition, but more specifically their understanding of the relationship between delta and epsilon. To explore the stability of students' knowledge across different contexts, we asked students about the temporal order of the two variables in three different contexts: dependence, control, and their *temporal order* (see the table below). Each individual interview lasted about 2 to 3 hours. These interviews were videotaped following recommendations in Derry et al. (2010).

Analysis

The first part of the analysis places students in categories based on their claim about the temporal order of epsilon and delta. There will be three categories: *epsilon first*, *delta first*, and *no order*. For a student to be classified into the category *epsilon first*, s/he would respond in the following way to the four questions. S/he would say that: 1) delta depends on epsilon; 2) one is trying to control x using delta, based on a given epsilon; 3) epsilon comes first and then delta; 4) the four variables are ordered in such a way where epsilon comes first then delta. For a student to be classified into the category *delta first*, s/he would respond in the following way to the five questions. S/he would say that: 1) epsilon depends on delta; 2) one is trying to control $f(x)$ using epsilon, based on delta; 3) delta comes first and then epsilon; 4) the four variables are ordered in such a way where delta comes first then epsilon. For a student to be classified as *no order*, there needs to be variance in responses across the different questions. In this study, we found few inconsistencies between the four different ways of asking the question.

The second part explores the range and nature of knowledge resources. We define *knowledge resources* as relevant prior knowledge that might be used to reason and justify the issue at hand. *Cued knowledge resources* are assertions students bring up as part(s) of a mechanism to justify a currently held position or opinion. We identify cued knowledge based on what students say in the moment. The analysis focuses on discussions around the four questions about the temporal order of delta. Reasonable interpretations for the statement will be considered and be put through the process of *competitive argumentation* (Schoenfeld, Smith & Arcavi, 1993) using other parts of the transcripts. With each of the cued knowledge resources, particular care will be given to investigate their origin and when it originally came up. Until there is consistent evidence of stance taken by a student, it would be impossible to make claims about the stability or how committed the student might be to the specific claim they made.

Results

Relationship Between the Epsilon and Delta

Five of seven students interviewed concluded that delta came first, 2 students concluded that epsilon came first and no student fell into the no order category. The table below shows the claims students made about the temporal order between epsilon and delta across the different contexts. We determine the student's final categorization by what the student said last about the relationship between epsilon and delta.

Student Initial	Question 15: Which depends on which?	Question 17: Which variable are you trying to control?	Question 18: Which one comes first?	Question 19: Order of [the four] variables.	Final categorization
DC	δ depends on ε	[Skipped]	N/A	N/A	Epsilon first
DL	ε depends on δ	δ you can control, ε you're trying to control.	N/A	N/A	Delta first
JJ	ε depends on δ	Control δ and [trying to] control ε.	N/A	N/A	Delta first
AD	δ depends on ε	[Skipped]	ε is first, you break down the epsilon to find delta.	Student decided not to try.	Epsilon first
DR	ε depends on δ	Trying to control δ so that you can get a smaller ε.	You get delta first then you get ε as your result.	$x, f(x), a, L, \delta, \varepsilon$	Delta first

SR	ε depends on δ	Trying to control x and $f(x)$, but not sure.	Calculate delta first, then used to calculate ε.	x, δ, $f(x)$, ε	Delta first
OB	ε depends on δ	Try to control x and δ to find ε.	Delta comes first, ε second.	δ, x, a, $f(x)$, ε	Delta first

Range and Diversity of Knowledge Resources

One very common knowledge resource that emerged in the pilot study was the output $f(x)$ was dependent on the input x. Students often associated this knowledge resource with that that argues that epsilon is a quantity related to $f(x)$ and delta is a quantity related to x. So $f(x)$ depends on x meant that epsilon must depend on delta, and so delta was first. Five of seven students used this fact to justify that epsilon depended on delta. DC cued these knowledge resources below.

> "Um [inaudible] well given that the, um, delta does generally or does seem to refer to the x value or the range of x values, the domain of x values that you want to be paying attention to, generally I think of functions, um, since a function is a relationship between dependent and independent variables, I tend to think of x as being you know as they are the, uh, independent variables. And so the y as being the ones that are altered by the x. So that's how you plug in numbers for functions, that's how you utilize functions in most cases. So it makes more sense to me to think that as epsilon being dependent on delta, where I'm assuming that delta is referring to x and epsilon is referring to y values" (turns 137-145).

DC reasoned that delta referred to a range of x-values and thus epsilon referred to a range of y-values, and since $f(x)$ or y depended on x, then it 'makes more sense to him' that epsilon depended on delta. So in this case we would argue that DC used the following knowledge resources to conclude that delta was first: 1) the dependence between x and $f(x)$; 2) delta refers to x values; and 3) epsilon refers to y values. Observe the similarity between what DC said with what DR said in her interview. "Um, see cus I was looking at it like the x or the $f(x)$ or the yeah, the $f(x)$ depends on the x and that's how I was like saying that epsilon depends on delta because epsilon like is related to the $f(x)$ or whatever" (turn 578). DR also relied on the dependence between x and $f(x)$, and she, too saw epsilon as "related to the $f(x)$." From the pilot studies other resources emerged from the data and we expect more to emerge as we interview more students. This analyses show the diverse knowledge resources students used to make sense of this relationship, and that most of these resources supported the assertion that epsilon depended on delta. No wonder students have a hard time with this relationship!

Conclusion and Broader Implications

This report shows that most students argue that within the formal definition, delta comes first. Students draw upon a range of resources, many of which support that claim the delta comes first. The remaining questions are the following. Once we recognize the range of resources that students use to make conclusions about the relationship between delta and epsilon, how do we begin to help students navigate through them? More specifically, how can we better assist students to refine or elaborate on the productive knowledge resources to make appropriate conclusions about the temporal order? A better understanding of the nature of these resources can facilitate the design of instruction that can help students bridge and reorganize these resources for a better understanding of the formal definition.

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