THE ROLE OF TIME IN A RELATED RATE SCENARIO

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Students who graduate with an engineering or science degree using applied mathematics are expected to synthesize concepts from calculus to solve problems. First semester calculus students attempting to understand the derivative as a rate of change encounter difficulties. Specifically, the challenges arise while making the decision to apply an average rate of change or an instantaneous rate of change (Zandieh, 2000) to the problem. This paper discusses how students view the derivative in an applied mathematical setting and investigates how the concept of time and other related quantities contribute to the development of a solution.

Key words: Calculus, rates of change, ladder problem, constant, time

Understanding the usage of the derivative and its related quantities is an essential component to applied mathematics, science, and engineering. This paper discusses ways in which students approach a solution to a standard calculus related rate problem. Generally in these problems, time is the independent variable. There are many quantities that involve time where time is explicitly or implicitly stated. The following literature describes student reasoning of the relationship of time to this calculus problem.

The ladder problem that Monk describes is directly related to applications of the derivative. Monk (1992) discusses Across-Time questions. *Across-Time* questions "ask the student to describe patterns of change in the value of a function that result from a pattern of change in the values of the input variables" (Monk, 1992, p. 176). Difficulties understanding an Across-Time view of functions arise from the students' grasp of relevant concepts. Once students were given a physical model, students are able to obtain correct answers.

Carlson (2002) defines "covariational reasoning to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, 2002, p. 354). Sophisticated covariational reasoning ability is important for representing functions graphically and understanding calculus (Thompson, 1994; Zandieh, 2000). Carlson (2002) presented a ladder problem which was a modification of the ladder problem reported by Monk (1992). Students were asked to represent a dynamic situation of a resting vertical ladder being pulled away at the bottom at a constant rate. In particular, they were asked to describe the speed of the top of the ladder as it slides down the wall. The student who had the correct response "performed a physical enactment of the situation, using a pencil and book on a table" (Carlson, 2002, p. 371).

Keene (2007) defines "dynamic reasoning as developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities to understand and solve problems." From data collected, a characterization of time as an explicit quantity was made. Time was used to reason both quantitatively and qualitatively.

Zandieh and Knapp (2006) discussed the role of metonymy in mathematical understanding of the derivative. When an interviewee was asked what the derivative was, it was referred to as a rate at which something increases. The student's vague description of rate of change made it difficult to determine if this change denoted the limit of the difference or if it referred to a qualitative rate of change.

The aforementioned literature is relevant to using time in developing solutions to applied mathematics problems. My analysis indicates an absence of time being emphasized as a contributing factor to the solution of the standard calculus ladder problem.

Methods

Four students from a second semester calculus for engineers class were interviewed in spring 2012, of which two are discussed in this paper. The interviews were semi-structured and the students were interviewed one at a time. The interviews lasted between twenty-five and forty minutes. The students were asked to use a Livescribe smartpen and notebook to record their thinking processes.

The students were given a 14-inch model which was to represent a 14-foot movable ladder. The wall and floor that supported this mobile ladder was the wall of a room and the top of the table was covered with butcher paper. They used this model ladder to answer the following question.

There are several terms that were observed in the data and coded. These terms were counted once per turn. Terms classified as rate included velocity, speed, and the appropriate usage of units. Time is explicitly stated with usage of units describing speed or rate. An average rate is defined as a change in distance per change in time. Velocity also adds a directional component to speed. Terms describing the change in distances were referred to as the change in the height of the ladder, drop, change in y, and increment.

Results

In this section, descriptions of what two of the four students discussed during the interviews are presented. George and Ted did not initially know how to handle the interview question and each asked the interviewer to clarify the interview question. George does not explicitly state time as he reasons through the question; however, he speaks of constant rate.

George

Before discussing any mathematical quantities that could be related to the problem, he instinctually answered the question. Clarifying questions followed his initial answer.

George: Does it speed up or slow down? I would say that it speeds up. [points to the top of the ladder and shows change in height by spreading his fingers out down along the wall from the top of the ladder]. Like, at the top of the ladder. Well, are we comparing as it [still handling the top of the ladder] relates to the bottom of the ladder? Or are we comparing it to the overall speed of the top of the ladder as it descends?

Notice as he initially discusses the solution to the problem, he discusses several quantities. The rates of the top and bottom of the ladder are mentioned right after he describes with his fingers the vertical drop of the ladder as compared to the bottom of the ladder. The phrase "relates to the bottom of the ladder" could refer to either the rate or the constant incremental distance that the ladder moves along the table.

He draws a picture illustrating the static situation of the ladder leaning up against a vertical wall. He discusses concepts of the derivative as a rate of change until he was suggested by the interviewer to create a table of measurements as the ladder was being pulled away from the base of the wall.

George: Ok, so as we're going down, it actually is accelerating. The increments are becoming greater and greater. So the answer to the question would be that it speeds up.

Interviewer: As evidenced by?

George: The measurements of it.

Once the model was used, words and phrases describing rate were replaced with words describing difference in heights.

Ted

In contrast to George, Ted does explicitly state time. After being asked the interview question from Figure 1, the student started to write in the notebook relating velocity, distance, and time. As he restates that the ladder is moving at a constant slow rate, he creates a graph and table in the notebook. He wanted to quantify the constant rate.

The interviewer inquired about Ted's thinking.

Ted: So I could just say that it's being pulled away a centimeter, once a centimeter per second. Cause velocity is distance over time. So if it's being pulled out in seconds. So the bottom of the ladder is being pulled out 1 centimeter per second.

[…]

Ted: So, I want to measure the time. I need a watch to measure the time but that might not be accurate.

Twelve minutes into the interview, he pulled the base of the ladder away from the wall in two inch increments and made a mark corresponding to the ladder's height on the vertical wall. He noted next to each tick mark on the wall the corresponding horizontal distance.

Ted: Ah HAH! So the top of the ladder slides down. It speeds up! When you pull it out at a constant rate, the top of the ladder speeds up. The top of the ladder [spreading his fingers out covering the distances between the marks he made on the wall], every two inches, would be greater.

Upon looking at the incremental y values, he determined that the ladder was speeding up. When asked to graph any quantities illustrating the problem, time was explicitly stated in his description of the movement of the bottom of the ladder.

Ted: Um, well if you had a way to correctly, um, to accurately pull the ladder out for time. You could use those variables to find, um, the velocity. So you would um, move the bottom of the ladder out with a time variable, um. And then measure how fast the top of the ladder, um, slides down.

The change in vertical distance indicated the ladder sped up as the base traveled a constant distance per unit of time.

Discussion

George and Ted both discussed rate and time explicitly when gathering information about the task. George expressed his image of the ladder's movement by relating the speed of the top of the ladder and the speed of the bottom of the ladder. The horizontal speed was affecting the vertical speed. Once George was prompted to use the model, words describing rate were replaced with horizontal and vertical changes.

Ted focused on the time it took for the bottom of the ladder to move at a constant rate. He wanted to record the time. When prompted to use the model to assist with visualization, he defined the constant rate to be one centimeter per second. He pulled the bottom of the ladder in two centimeter increments and recorded the heights of the ladder.

Ted's measuring of fixed distances representing a unit of time can be considered a speed-length. A speed-length is defined as the "distance traveled in one unit of time" (Thompson, 1994; Thompson & Thompson, 1992).

George and Ted changed their language of rate and time to changes in vertical and horizontal distance when offered the manipulative. Reasons for this require further investigation.

How do the theoretical frameworks mentioned above connect with the data seen? How might the interviews be structured to understand why the language of rate and time changed to that of differences in distances?

A 14 – foot ladder is being pulled away from a wall at a constant, slow rate. Does the top of the ladder slide down the wall at a constant rate? Or, does it speed up or slow down?

Figure 1. Interview Question

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