ILLUSTRATING A THEORY OF PEDAGOGICAL CONTENT KNOWLEDGE FOR SECONDARY AND POST-SECONDARY MATHEMATICS INSTRUCTION

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Abstract. The accepted framing of pedagogical content knowledge (PCK) as mathematical knowledge for teaching has centered on the question: What mathematical reasoning, insight, understanding, and skills are required for a person to teach mathematics? Many have worked to address this question, particularly among K-8 teachers. What about teachers with broader mathematics knowledge (e.g., from algebra to proof-based understandings of topics in advanced mathematics)? There is a need for examples and theory in the context of teachers with greater mathematical preparation and older students with varied and complex experiences in learning mathematics. This theory development piece offers background and examples for an extended theory of PCK as the interplay among conceptually-rich mathematical understandings, experience of teaching, and multiple culturally-mediated classroom interactions.

Keywords: Pedagogical content knowledge, Discourse, Intercultural awareness

Since Shulman's (1986) seminal work, a rich collection of theories and measures of mathematics pedagogical content knowledge (PCK) continues to grow (e.g., Hill, Ball, & Schilling, 2008; Silverman & Thompson, 2008). As PCK has become widely utilized in research on early grades (K-8) teacher development, a model based on mathematical knowledge for teaching (MKT) has emerged (Hill, Blunk, et al., 2008). Most framing of MKT for early grades includes little in the way of intermediate and advanced algebra or of proof-based understandings, such as are found in college mathematics. The existing K-8 work is valuable in thinking about PCK in secondary and post-secondary settings and to build on it, there is a need for examples and theory in the context of teachers with greater mathematical preparation and older students with varied and complex experiences in learning mathematics.

The common framing of mathematical knowledge for teaching has centered on the question: What mathematical reasoning, insight, understanding, and skills are required for a person to teach mathematics? Many have worked to develop measures to address this question, most notably Ball and colleagues (Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). In their work they have defined three types of PCK: knowledge of curriculum, knowledge of content and students (KCS), knowledge of content and teaching (KCT). Even with this carefully developed model, challenges exist in identifying and measuring PCK (Hill, et al., 2008).

Other researchers have offered a supplement to the K-8 view, emergent from radical constructivist perspectives (i.e., Piagetian). It is the idea that for some, PCK is "predicated on coherent and generative understandings of the big mathematical ideas that make up the curriculum." (Silverman & Thompson, 2008, p. 502). In this framing, PCK grows when a teacher gets better at the transformation of personal and intimate forms of mathematical knowing. Our purpose in building theory is to describe and illustrate an unpacking of this idea while also attending to the reality of culturally heterogeneous classroom contexts.

Here we report on our efforts to develop an expanded theory and model of PCK that attends to a key aspect of Shulman's framing of PCK that is absent in existing models: a fourth component of mathematical knowledge for teaching, *discourse knowledge*. This brings to PCK the mathematical semiotics that was part of Shulman's original description. Ultimately, we seek to develop a theory and measurement tools/guidelines that allow exploration of such questions as: *What is the interplay among mathematical understandings, experience of teaching, and culturally-mediated communication in defining and growing algebra PCK? ...proof PCK?*

We start with brief definitions associated with "discourse," describe our model with additional PCK constructs, and make a foray into some key ideas in intercultural orientation. Over the last 10 years, the authors have been involved in a variety of ways in research and professional development with college and university faculty, in-service secondary mathematics teachers, and their students. In that work, we are regularly asked by mathematically-trained stakeholders for examples and non-examples of PCK in use. To provide a compact and relatively simple contextualized illustration, we conclude with two classroom examples. Vignette 1 represents Teacher Pat in the third year of teaching experience; Vignette 2 represents Pat's classroom again, after three more years that included professional work related to responsive noticing of student thinking for generating and sustaining conceptually-focused discourse during instruction. The two vignettes and brief analyses of them are presented to illustrate the theorized PCK constructs. These illustrations are *not* definitions. They are offered as anchors for discussion. For this initial report we include a pair of algebra-based vignettes. At the RUME session and in the final long paper, we will share similar snapshots and analyses for at least one advanced topic (e.g., linear algebra or group theory). Our proposed framework relies on three existing theories related to human interaction in mathematics teaching and learning: for discourse, for PCK, and for intercultural sensitivity development.

Background on Discourse

A classroom culture is a set of values, beliefs, behaviors, and norms shared by the teacher and students that can be reshaped by the people in the room (Hammer, 2009). Though not everyone in the classroom may describe the culture in the same way, there would be a general center of agreement about a set of classroom norms, values, beliefs, and behaviors. Gee (1996) distinguishes between Discourse and discourse. The "little d" discourse is about language-in-use (this may include connected stretches of utterances and other agreed-upon ways of communicating mathematics such as symbolic statements or graphs). Discourse ("big D") includes little d discourse but also includes other types of communication that happen in the classroom (e.g., gestures, tone, pitch, volume, and preferred ways of presenting information). Big D Discourse also includes syntactic knowledge, an aspect of PCK introduced by Shulman and colleagues but not regularly or explicitly tackled in current K-8 focused approaches to describing and measuring PCK; understandings about how to conduct mathematical inquiry. Gee notes:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and 'artifacts', of thinking, feeling, believing, valuing, and acting that can be used to identify oneself as a member of a socially meaningful group or 'social network', or to signal (that one is playing) a socially meaningful 'role' (p. 131).

The forms of communication in discourse are usually explicit and observable, while the culturally embedded nature of inquiry in Discourse is largely implicit. That is, as part of PCK, this is knowledge for working effectively in the classroom with the multiplicity of Discourses students bring into the classroom. In particular, each Discourse includes a cultural context. Discourses may differ from person to person or group to group. The ways that teachers and learners are aware of and respond to these multiple cultures is a consequence of their orientation towards cultural difference, their *intercultural orientation* (individually and collectively). A teacher with rich knowledge of multiple Discourses, whose orientation towards cultural difference is developmentally advanced, can juggle and balance and engage with myriad cultures in-the-moment to support effective communication among all in the room. We come back to intercultural orientation, below, after unpacking what we mean by Discourse a bit more.

The "big D" Discourse of academic mathematics values particular kinds of "little d" discourse. Valued inscriptions are figural (e.g., in representations such as graphs of functions) and logico-deductive (e.g., proofs). Especially valued in advanced mathematics are explanation, justification, and validation (Arcavi, Kessel, Meira, & Smith, 1998; DeFranco, 1996; Weber, 2004). As in other fields, instructors ask questions to evaluate what students know and to elicit what students think. For instance, a model of classroom interaction common in the U.S. is the dialogic pattern where teacher *initiates* – student *responds* - and teacher *evaluates* (Mehan, 1979). More recent work has led to a more broadly defined *initiation –response –follow-up* or *I•R•F* structure (Wells, 1993). In college classrooms, this is most often initiated by teachers, but not exclusively so, and the (implicit) rules for how initiating, responding, and following-up will happen are worked out by the people in the room (Nickerson & Bowers, 2008). In his ethnographic work, Mehan identified four types of teacher questions (see Table 1).

Table 1. Initiate – Respond – Follow-up (*I•R•F*) question types and anticipated response types.

Evaluate what students know	***Elicit what students think***
Choices – response constrained to agreeing or not with a statement (e.g., Did you get 21?) ***Products*** – response is a fact (e.g., What did you get?)	***Processes*** – response is an interpretation or opinion (e.g., Why does 21 make sense here?) ***Metaprocesses*** – response involves reflection on connecting question, context, and response (e.g., What does the 21 represent? How do you know?)

Research suggests that U.S. mathematics instructional practice lives largely to the left of Table 1 (Stigler & Hiebert, 2004; Wood, 1994). The unfortunate aspect here is not necessarily the fact that evaluative questions are common but that the eliciting *processes* and *metaprocesses*, in the right column, are not. These more complex spurs for discourse can lead to iterative patterns that cycle through and revisit the frame of reference "in ways that situate it in a larger context of mathematical concepts" and foster "mathematical meaning- making" (Truxaw & DeFranco, 2008, p. 514). The use of process and metaprocess questions, for example as *follow-up (F)*, readily expands discourse into the "reflective toss" realm of comparing and contrasting different ways of thinking (with justification but without judgment), monitoring of a discussion itself, as well as attending to the evolution of one's own thinking (van Zee & Minstrell, 1997).

Four Component Model of Pedagogical Content Knowledge

While Hill, et al. (2008) acknowledge the importance of teacher knowledge of standard and non-standard mathematical representations and communication, discourse knowledge as we construe it – composed of both discourse and Discourse understandings – does not appear explicitly in their model of pedagogical content knowledge. One way of visualizing our extension is as a tetrahedron whose base is the MKT model with apex of discourse knowledge (see Figure 1, next page). As indicated in Figure 1, our attention has focused on discourse knowledge and the three "edges" connecting it to the components in the MKT model (Hauk, Jackson, & Noblet, 2010). These edges, though labeled in Figure 1 and discussed here as kinds of "knowledge," might more appropriately be labeled as "ways of thinking," with the aspects of the MKT model taken as "(ways of) understanding" (Harel, 2008). The distinction is still an area of theory development for the authors and will form part of the conference session discussion.

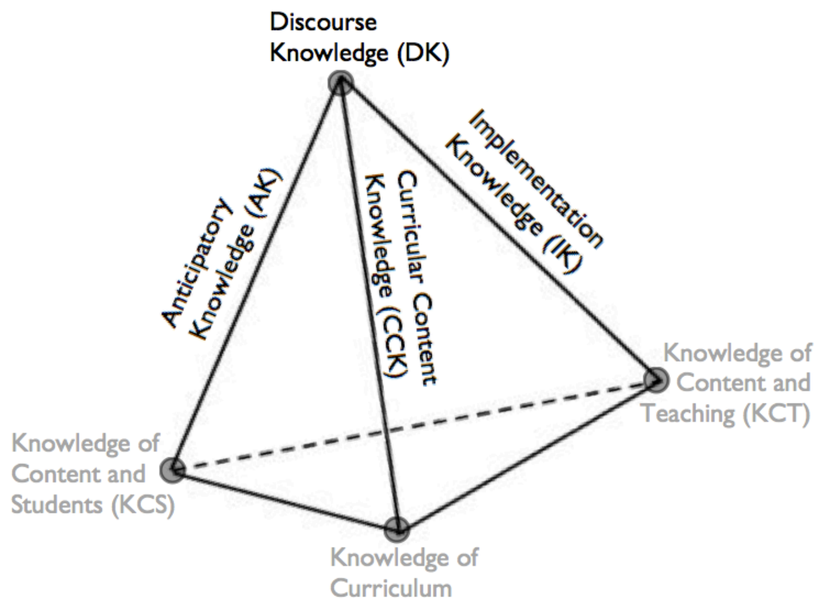

Figure 1. Tetrahedron to visualize relationship among PCK model components. Corners of the base are the aspects of MKT PCK articulated in Hill, Ball, and Schilling (2008).

As indicated above, ***Discourse knowledge*** (**DK**) is d/Discourse knowledge about the culturally embedded nature of inquiry and forms of communication in mathematics (both in and out of educational settings). This collection of understandings includes syntactic knowledge, "of how to conduct inquiry in the discipline" (Grossman, Wilson, & Shulman, 1989, p. 29).

Curricular content knowledge (**CCK**) is substantive knowledge about topics, procedures, and concepts along with a comprehension of the relationships among them and conventions for reading, writing, and speaking them in school curricula. In its most robust form, this part of PCK contributes to what Ma (1999) called "profound understanding of mathematics" (p. 120). In combination, curricular content and discourse knowledge are the home of Simon's (2006) "key developmental understandings."

Anticipatory knowledge (**AK**) is an awareness of, and responsiveness to, the diverse ways in which learners may engage with content, processes, and concepts. Part of anticipatory knowledge growth involves what Piaget called "decentering" – building skill in shifting from an ego-centric to an ego-relative view for seeing and communicating about a mathematical idea or way of thinking from the perspective of another (e.g., eliciting, noticing, and responding to student thinking; Carlson, Moore, Bowling, & Ortiz, 2007). Teachers with rich anticipatory knowledge know how to manage the tensions among their own instrumental and relational understandings of mathematics and its learning and those of their students (Skemp, 1976). Such perspective-shifting is deeply connected to Discourse through the awareness of "other" as different from "self." We return to this idea, below, in discussion of intercultural orientation.

Implementation knowledge (**IK**) is about how to enact teaching intentions in the classroom. Moreover, for us, it includes how to adapt teaching according to content and socio-cultural context and act on decisions informed by discourse as well as curricular content and anticipatory ways of thinking. We do not argue for an intention to enculturate in the sense of Kirshner's (2002) "teaching as enculturation" (i.e., to identify a reference culture and then target instruction for students to acquire particular dispositions). Nor do we propose his alternate framings (habituation, construction) or any other preference for implementation knowledge paradigm.

Intercultural Orientation

The construct of "big D" Discourse as part of mathematics PCK pivots on the idea of intercultural orientation. Our referent framework is the *Developmental Model of Intercultural Sensitivity* (Bennett & Bennett, 2004). The developmental continuum of orientations towards awareness of cultural difference, of "other," runs from a monocultural or ethnocentric "denial" of difference based in the assumption "Everybody is like me" to an intercultural and ethnorelative "adaptation" to difference. The development from denial to the "polarization" orientation comes with the recognition of difference, of light and dark in viewing a situation (e.g., Figure 2a).

Figure 2a.

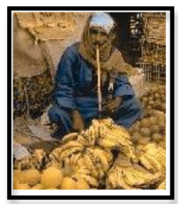

Figure 2b.

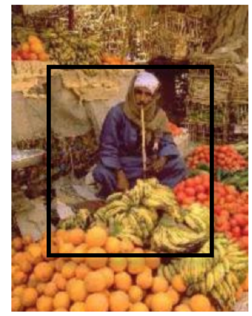

Figure 2c.

Denial ⇨ Polarization ⇨ Minimization ⇨ Acceptance ⇨ Adaptation

Figure 2. Intercultural orientations and developmental continuum.

The polarization orientation is driven by the assumption "Everybody should be like me/my group" and is an orientation that views difference in terms of "us" and "them." Evaluative prompts about student thinking are more likely (left side of Table 1) for this orientation. Moving along the continuum towards ethno-relative perspectives leads to a minimizing of difference, focusing on similarities, commonality, and presumed universals (e.g., biological similarities – we all have human brains so we all learn math essentially the same way; and values – we all know the difference between right and wrong and naturally will seek right). This is the "minimization" orientation. A person with this orientation will be blind to recognition and appreciation of subtleties in difference (e.g., Figure 2b, a "colorblind" view). The minimization orientation tends to take the form of ignoring fine detail in how people might have differing ways of thinking. For example, efforts at eliciting d/Discourse (right side of Table 1) may take the form of listening for particular ways of thinking. Transition from a minimization orientation to the "acceptance" orientation involves attention to nuance and a growing awareness of self and others as having culture and belonging to cultures (plural) that may differ in both obvious and subtle ways. While aware of difference and the importance of relative context, how to respond and what to respond *in the moment of interaction* is still elusive. From this orientation, classroom d/Discourse may include process and metaprocess prompts, but sustained cycles of such interactions can be challenging to maintain in the immediacy of dynamic classroom conversation. The transition to "adaptation" involves developing frameworks for perception, and responsive skills, that attend to a spectrum of detail in an interaction (e.g., the detailed and contextualized view in Figure 2c). Adaptation is an orientation where one may shift cultural perspective, without violating one's authentic self, and adjust communication and behavior in culturally and contextually appropriate ways. There is an instrument for measuring developmental orientation (see idiinventory.com). One area of ongoing work for us is the relationship between intercultural orientation and what orientation(s) may be necessary, if not sufficient, for rich d/Discourse knowledge.

Vignettes and Discussion

Vignette 1–Snapshot of a Classroom

ALGEBRA CLASSROOM*

PAT (teacher) stands in front of whiteboard, 36 students in 6 rows of 6 face the front in small desk-chairs. Problem on the board reads: *At some time in the future John will be 38 years old. At that time he will be 3 times as old as Sally. Sally is now 7 years old. How old is John now?* Shuffling of paper and scratching of pencils but no voices as students work.

Pat: Okay. Let's talk about this. What did you get? How is it that you thought about it?

Lee: I divided 38 by 3. Then I subtracted 7 from 12 2/3 and got 5 2/3. Then I subtracted that from 38 and got 32 1/3. *(Pause)* John is 32 1/3.

Pat: Right. *(Pause)* Why did you divide 38 by 3?

Lee: *(Appearing puzzled by the question, Lee looks back at her work. She looks again at the original problem.)* Because John is 3 times older.

Pat: Okay. *(Looks around the room)*. Any questions? *(Turns and erases the board).*

Jackie (quietly to herself): Isn't the answer 21?

Pat: Hum? *(Turns to face the room)* No. If he was 21 he'd be three times as old as Sally is *now*.

Jackie: It says that he is 3 times as old as Sally, and Sally is 7.

Pat: Well, the problem says John is 3 times as old when John is 38, at some time in the future. *(Pause)* Do you understand?

Jackie: (shrugging): Okay.

Pat: Okay. Does anyone have any more questions? Okay. Now you try one, number 19 on page 33.

Problem 19. *At some time in the past Luana was 24 years old. At that time she was 4 times as old as Rodney. Rodney is now 12 years old. When will Luana be 40?*

*This and the content of Vignette 2 are derived from Thompson, Philip, Thompson, & Boyd (1994).

Figure 3. Vignette representing Teacher Pat's classroom instruction in third year of teaching.

Curricular Content Knowledge. In Vignette 1, Teacher Pat identifies Lee's work as correct. However, the vignette does not offer detail on Pat's *curricular* content knowledge. In his inference that it is similar to the newly assigned item, Pat only implicitly connects the problem to other mathematics relevant to the students in the local context of their learning,. In fact, there are important differences between the mathematical thinking needed for the John and Sally problem and the level-appropriate ideas likely to be called on to tackle the new Luana and Rodney item. Vignette 2 (next page) provides more insight into what the teacher notices about student thinking. Pat demonstrates knowledge of the mathematical requirements appropriate to the curricular focus of the class. Pat's attention to the multiple problem solving approaches and acting as a guide through the discussion are evidence that the particular concepts, and the use of particular tools (e.g., the table) are curricular-level-appropriate for the class.

Anticipatory Knowledge. In Vignette 1, Pat demonstrates anticipatory knowledge (and approval) of a correct student solution path when expressed as procedural knowledge. A moment later, though, Pat appears unaware of the origin or nature of Jackie's confusion. That is, in Vignette 1, Teacher Pat does not appear to anticipate common student struggles (unpacking mathematical relationships from densely worded problems and organizing information from a word problem context). As we see in Vignette 2, such anticipation could be a valuable resource for enhancing students' understanding of mathematics. In the second vignette, Teacher Pat anticipates that students may focus on steps to the right answer and inserts a purposeful halt. Pat aims for a socio-mathematical norm in which explanation for sense making regarding how and why of doing mathematics is valued. In addition, Pat elicits an intellectual need for considering the potential mismatch of the information from students through a display of multiple representations – table and number sentences. In the second scenario, Teacher Pat has a richer anticipatory knowledge. It leads to broader student contribution, allows for the teacher to make sense of students' current thinking, and helps sustain engagement of students in the lesson.

Vignette 2 - Snapshot of a Classroom
ALGEBRA CLASSROOM

PAT stands in front of whiteboard, 36 students in 6 rows of 6 face the front in small desk-chairs. John and Sally problem is on the board. Shuffling of paper, scratching of pencils, whispers as students work.

Pat: Let's talk about this problem a bit. How did you think about the information in it?
Sam: Well, you gotta start by dividing 38 by 3. Then take away-
Pat: *(Interrupting)* Wait! Before you tell us about the calculations you did, explain to us why you did what you did. What were you trying to find?
Sam: Well, you know that John is 3 times as old, so you divide 38 by 3 to find out how old Sally is.
Pat: Do you all agree with Sam's thinking?
Several students say, "Yes"; others nod their heads.
Lia: That's not going tell you how old Sally is *now*. It'll tell you how old Sally is when John is 38.
Pat: Is that what you had in mind, Sam?
Sam: Yes.
Pat: *(To the class)* What does the 38 stand for?
Lia: John's age in the future.
Pat: So 38 is not how old John is now. It's how old John will be in the future. *(Pat starts a table on the board, as discussion continues, he adds to it)*

	Future	
John	38	

The problem says that when John gets to be 38 he will be 3 times as old as Sally. Does that mean "3 times as old as Sally is *now*" or "3 times as old as Sally will be when John is 38?"
Several students respond, "When John is 38."
Jonah: Couldn't you just say John is 21? *(Pause)* Couldn't you just multiply 3 times 7?
Pat: What will that give you?
Jonah: Twenty-one!
Pat: But what would 21 represent? What is it that's 21?
Jonah: That's how old John is now. Isn't that what we want to find?
Maura: No! *(Pause)* I mean, yes! That's what we want to find, but that's not right!
Pat: What is it that is not right, Maura? We do want to find out how old John is now, don't we?
Maura: Right. But see, he's not 3 times older than Sally now. He'll be 3 times older than Sally *when he is 38*. You have to keep track of what's true now and what's so in the future.*(Pat adds to the table)*
Pat: Okay, so how are we going to use the information that John will be 3 times as old as Sally when he gets to be 38? *(Pause)* Who can explain?
Sam: You can divide 38 by 3 and get 12.66....
Pat: Remember to tell us what your numbers stand for. What does the 12.66... stand for?
Gina: That's how old Sally will be.
Pat: When? *(Several respond, "When John is 38.")*
Pat: (Looking around) Let's keep going. Nasir?
Nasir: Okay, you can say that Sally will be 12ish. So if you subtract 7 from that you get 5. Then you take away 5 from 38 and get John is 33. Done!
Pat: Wait a minute, too fast! Explain your reasoning.
Nasir: (Patiently) You know Sally will be 12 and something, and you know that she is 7 now. So that means that there are 5 years between now and then. Actually a little more than 5 years.
Pat: So 5 years is how much time there is between now and the time in the future when John is 38?
Nasir: Yes. So if you take 5 away from 38, that's how old John is now.
Pat: Did everyone follow that? (Pause) Who will recap the solution we've just been through?

Figure 4. Vignette representing Teacher Pat's classroom instruction in sixth year of teaching.

Discourse Knowledge. In Vignette 1, Teacher Pat foregrounds the correct answer and a single path to that answer. That is, the primary discourse (little "d") in the classroom is largely univocal: Pat's utterances to identify a correct procedure. Discourse (big D) is also centered with the teacher, as the explanations valued in the classroom are Pat's. In Vignette 2, Pat asks students to "explain to us why you did what you did, what were you trying to find?" To be able to participate in discourse (little "d"), responding students have been asked explicitly to offer their own thinking to provide a convincing argument. Such eliciting questions by Pat are evidence of an intention to build a particular socio-mathematical norm. An aspect of the Discourse, then, is that engaging in deep explanation is an expectation of all in the classroom. Further evidence is Pat's contribution of a table to organize information as well as in the final question asked. While Pat's voice is first to offer the table, the utterances in the room are dialogic, not univocal.

Implementation Knowledge. In Vignette 1, Teacher Pat implements *choice* and *product* questions. If these questions dominate a teacher's contributions to discourse, then multiple disconnected *I•R•F* interactions can yield a teacher-regulated kind of interaction that does not include deep participation by students. This can be true even in inquiry-based instruction (Nassaji & Wells, 2000; Wertsch, 1998). This is evident from Pat's responses and questions in which the focus is on the steps of the computation rather than reasoning. There is no student-to-student interaction and when Pat overhears Jackie's question to herself, Pat responds by correcting (staying to the left of Table 1). In the second vignette, Pat elicits student thinking. The environment of the classroom is interactive with students sharing their reasoning and questioning each other as Pat encourages them to make sense of each other's ideas. As students present their ideas, Pat emphasizes the process rather than the product. Pat also uses multiple modes of discourse, including a table and confirming questions in order to support the needs of various students. Pat ensures that the students use mathematical terminology and language as they present their solution and share their understanding.

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