UNDER THE RADAR: FOUNDATIONAL MEANINGS THAT SECONDARY MATHEMATICS TEACHERS NEED, DO NOT HAVE, BUT COLLEGES ASSUME

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High school mathematics teachers must have coherent systems of mathematical meanings to teach mathematical ideas well. One hundred five teachers were given a battery of items to discern meanings they held in with respect to quantities, variables, functions, and structure. This paper reports findings on a sample of items that, by themselves, should alert college mathematics professors that foundational understandings they assume students have in advanced mathematics courses likely are commonly missing.

Key words: Mathematical meanings, mathematical knowledge for teaching, undergraduate mathematics, mathematics teacher education

In this paper we report partial results from a project that attempts to discern teachers' *mathematical meanings for teaching secondary mathematics* (MMTsm). The project's focus is on assessing mathematical meanings teachers have, meanings they attempt to convey in instruction, and relationships between them (Thompson, in press). For the present purpose, however, we focus on results of an assessment given in the summer of 2012 to 105 secondary mathematics teachers in two states in terms of their implications for teachers' undergraduate mathematical preparation.

The assessment consisted of 36 paper items and 3 animation items. We will not discuss the animation items here. The 36 paper items were distributed among three forms, each form consisting of 9 items that were common to all forms and 9 items unique to each form.

We administered the assessment to 141 teachers in June and July of 2012. The teachers were in four groups: Three groups (126 teachers) from an MSP project conducted at a major Midwestern university and one group (15 teachers) from an MSP project conducted at a major Southwestern university. One hundred five (105) of these teachers were currently teaching high school mathematics or were starting to teach in Fall 2012. Table 1 gives a breakdown of teachers' teaching experience (number of courses taught, "Crs Tgt") in relation to formal mathematical preparation.

Table 1. Teachers' experience

	Major			
Crs Tgt	***Math***	***MathEd***	***Other***	***total***
0	1	5	0	6
1-5	7	2	10	19
6-10	6	3	11	20
11-20	5	15	8	28
>20	10	18	4	32
total	29	43	33	105

The items were drawn from the areas of variables and variation (4), covariation (4), functions (7), proportionality (5), rate of change (7), and structure (9). No item that has face validity to teachers can focus on just one these areas. Items therefore have aspects of two or more areas but a greater reliance on one. The item's category is our estimation of the most prominent meaning involved.

In this paper we will discuss results from three specific items: one on function, one on structure, and one on rate of change.

Function Item

The function item is given in Figure 1. It was given to 34 teachers. The item draws upon a scheme of meanings entailed in the use of function notation, namely that, for example, "*w*" is the function's name, "*u*" represents an input value, and that "*w*(*u*)" represents the output value that is determined by the rule "sin(*u* – 1) if *u* ≥ 1" when a value of *u* is given as input. The item also entails an additional aspect of a process conception of a function definition (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992): teachers need to envision a process whereby values of variables are "passed" from one function definition to another.

Here are two function definitions.

$$w(u) = \sin(u-1) \text{ if } u \geq 1$$
$$q(r) = \sqrt{r^2 - r^3} \text{ if } 0 \leq r < 1$$

Here is a third function *c*, defined in two parts, whose definition refers to *w* and *q*. Place the correct letter in each blank so that the function *c* is properly defined.

$$c(v) = \begin{cases} q(\_\_) \text{ if } 0 \leq \_\_\_ < 1 \\ w(\_\_) \text{ if } \_\_\_ \geq 1 \end{cases}$$

Figure 1. Item F07v1: Understanding function notation.

The letter "v" should be placed in each blank because *v* represents the value of the function's input, and the right hand side gives a rule for how to produce *c*'s output when given a value of *v*.

Table 2 shows that only 9 of 34 teachers (26%) wrote *v* in the blanks, and that 17 of 34 teachers (50%) wrote *r* in the blanks associated with *q* and wrote *u* in the blanks association with *w*. Table 2 also shows that only 1/3 of teachers with a BS in Math or Math Ed wrote *v* in the blanks of *c*'s definition. Others either wrote *r* and *u* or a combination of *r*, *u*, and *v*. Table 3 shows that teachers who had taught precalculus, calculus (AB or BC), or differential equations were not likely to have written *v* in the slots in *c*'s definition.

Table 2. Responses for Item F07v1

	Math	*MathEd*	*Other*	*total*
R U	5	5	7	17
V	2	5	2	9
Mix	0	2	2	4
I don't know	0	1	1	2
No Answer	0	2	0	2
total	7	15	12	34

Table 3. Responses to F07v1 by Precalculus+ teachers

	Math	*MathEd*	*Other*	*total*
R U	2	2	0	4
V	0	1	2	3
I don't know	0	1	1	2
Mix	0	1	0	1
total	2	5	3	10

Teachers' responses on other items in the function group shed light on their scheme for function notation. It is that many teachers think of function notation idiomatically. As an idiom, the constituent elements of "*w*(*u*)" have no direct relationship to the meaning of "*w*(*u*)". Rather, the idiom's meaning is figurative, and is made through the use of the expression in its entirety. It is as if "*w*(*u*)" is the function's name, whereby a teacher writing something like "$w(u) = 3x + 5$" is expressing something like "the function named '$w(u)$' is $3x + 5$".

Structure Item

An important aspect of seeing mathematical structure is to see something complex as also being something simple, and to see something simple as entailing an internal complexity. The structure item in Figure 2 requests teachers either to see four terms as three terms or two terms as three terms.

> **Δ is a binary operation over the real numbers. It is associative. This means:**
>
> **For all real numbers a, b, and c, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.**
>
> **a) Let u, v, w, and z be real numbers. Can the associative property of Δ be applied to the expression below? Explain.**
>
> $$(u \Delta v) \Delta (w \Delta z)$$
>
> **b) Why might a teacher ask this question?**

Figure 2. Structure item S01v3.

Viewed structurally, the expression $(u \Delta v) \Delta (w \Delta z)$ can be seen as

$$\overbrace{(u \Delta v)}^{a} \Delta (w \Delta z)$$

or as

$$(u \Delta v) \Delta \overbrace{(w \Delta z)}^{c}.$$

In either case, the associative property of Δ can be applied to the resulting 3 terms.

Table 4 shows that 16 of 111 high school math teachers (5 Math, 5 MathEd, and 6 Other) said that the associative property of Δ cannot be applied to $(u \Delta v) \Delta (w \Delta z)$. The most common reason was that associativity requires 3 terms and $(u \Delta v) \Delta (w \Delta z)$ has either two or four terms. Table 5 shows that 74 of 111 high school math teachers said "yes", that the associative property of Δ could be applied to the expression $(u \Delta v) \Delta (w \Delta z)$. Of those 74 teachers, 16 gave a valid explanation that was either a correct application of associativity or a statement that two terms could be considered one. Fifty-eight (58) teachers said "yes", that the associative property of Δ can be applied to the expression $(u \Delta v) \Delta (w \Delta z)$, and then gave an invalid demonstration, a non sequitur justification, or no justification.

Table 4. Explanations by High School Math Teachers' Who Said "No"

Explanation	***Math***	***MathEd***	***Other***	***total***
Associative property requires 3 elements	2	2	3	7
Interpreted problem as about something other than associativity	2	0	2	4
Imported the properties of an arithmetic operation	1	1	0	2
No justification	0	1	1	2
Used commutativity	0	1	0	1
total	5	5	6	16

Table 5. Explanations by High School Teachers' Who Said "Yes"

Explanations by teachers who answered "Yes"	*Math*	*MathEd*	*Other*	*total*
Imported the properties of an arithmetic operation	3	6	8	17
Explained that (uv)(wz) can be thought of as a(wz), where a=(uv), and associativity of Δ applied to it	5	11	0	16
You said in the item's stem that Δ is associative	4	5	5	14
Parentheses don't matter for associativity	5	4	4	13
No justification	5	4	0	9
Used commutativity	1	1	2	4
Interpreted problem as about something other than associativity	0	0	1	1
total	23	31	20	74

Other items in the Structure group clarify teachers' difficulty with S01v3. Together they suggest that many teachers' thinking is constrained to one level of organization—that they see complex mathematical statements as unstructured strings. If this is indeed the case, then it is understandable that teachers with this way of thinking are challenged by sophisticated concepts—concepts having nested levels of entailed meanings—and by complex mathematical statements.

Rate of Change

Rate of change, which entails both ideas of relative change and ideas of accumulation, is *the* foundational concept in the calculus (Thompson, 1994a; Thompson, 1994b). Developing students' understandings of rate of change is a primary task of secondary school mathematics instruction. Teachers without a rich scheme of meanings for rate of change will be limited in helping students support a rich scheme of meanings. The rate item R08v1 appears in Figure 3. It is a standard algebra question, usually included under the heading "weighted averages". The key to reasoning to a solution is to understand that a round trip will take 3 hours (180 miles at 60 mi/hr), and that the first part takes 2.25 hours (90 mi at 40 mi/hr), leaving 0.75 hours to travel the returning 90 miles. Thus, the car would need to have an average speed of 120 mi/hr to have a round-trip average speed of 60 mi/hr.

> A car went from San Diego to El Centro, a distance of 90 miles, at 40 miles per hour. At what speed would it need to return to San Diego if it were to have an average speed of 60 miles per hour over the round trip?

Figure 3. Rate of change item R08v1.

Thirty-four teachers received item R08v1. Table 5 gives teachers responses. Only 7 of 34 teachers (21%) answered correctly. The most common response was 80 mi/hr. Teachers' work made clear that they arrived at 80 by solving the equation $(x + 40)/2 = 60$. We hoped that teachers who had taught precalculus, calculus (AB or BC), or differential equations would have a higher success rate. Table 6 shows a success rate of 40% among these teachers. While it is higher than other teachers, it is still surprisingly low.

Table 6. Responses to R08v1.

	Math	*MathEd*	*Other*	*total*
80 mi/hr	4	9	3	16
120 mi/hr	2	2	3	7
Analyst could not interpret	0	1	4	5
No answer—no written work	1	1	2	4
Computed time for first trip	0	2	0	2
total	7	15	12	34

Table 7. Responses to R08v1 by Precalculs+ teachers

	Math	*MathEd*	*Other*	*total*
80 mi/hr	1	2	1	4
120 mi/hr	1	1	2	4
Computed time for first trip	0	1	0	1
No answer—no written work	0	1	0	1
total	2	5	3	10

Other items in the Rate group shed light on teachers' meanings for rate of change. In essence, their scheme involves only one quantity – rate, and that quantity itself is more a number than a quantity. A mature scheme of meanings for rate of change involves three quantities – two quantities changing simultaneously and a third quantity (rate) that entails a multiplicative relationship between them. If this is correct (it is consistent with other research on students' understanding of the calculus), then these teachers had these meanings as students of calculus and their students will have a high probability of having a one-quantity meaning of rate when they enter calculus.

Discussion

Our research suggests that a significant percentage of teachers have meanings and schemes of meanings that are poorly developed. We see two possibilities, neither of which speak well of university mathematics education. Either teachers developed these weak meanings as undergraduate students, or they developed many of these meanings when they were high school students and they carried these meanings throughout their undergraduate mathematics coursework. In either case, they assimilated their undergraduate mathematics instruction into these schemes and their understanding of undergraduate mathematics instruction was built upon these schemes. We hope that by making college mathematics instructors aware of how fragile the base of understanding is among students they are teaching, that mathematics departments will become proactive in adjusting their introductory mathematics curriculum and instruction.

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