ON THE SENSITIVITY OF PROBLEM PHRASING – EXPLORING THE RELIANCE OF STUDENT RESPONSES ON PARTICULAR REPRESENTATIONS OF INFINITE SERIES

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This study will demonstrate the ways in which students' ideas about convergence of infinite series are deeply connected to the particular representation of the mathematical content, in ways that are often conflicting and self-contradictory. Specifically, this study explores the different limiting processes that students attend to when presented with five different phrasings of a particular mathematical task - $\sum(1/2)^n$ *- and the ways in which each phrasing of the task brings to light different ideas that were not evident or salient in the other phrasings of the same task. This research suggests that when attempting to gain a more robust understanding of the ways that students extend the ideas of calculus – in this case, limit – one must take care to attend to not only students' reasoning and explanation, but also the implications of the representations chosen to probe students' conceptions, as these representations may mask or alter student responses.*

Keywords: Calculus, Infinite Series, Interviewing, Tasks

Background and Framing

Existing literature on students' understanding of infinite series (i.e. series of numbers, Taylor series, power series) is extremely sparse, despite the overwhelming notion that it represents (1) the most important topic that students can understand from traditional second semester calculus, if they are preparing for a future in engineering, physics, and other related fields (e.g. Alcock & Simpson, 2009; Tall & Schwarzenberger 1978), and (2) the topic with which students have the most difficulty, when considering the entirety of the traditional second semester calculus syllabus (e.g. Monaghan 2001; Biza, Nardi, & Gonzalez-Martín 2008; this author, in preparation). Literature suggests an unbalanced treatment of the related material, as "sequences are played down, or even omitted, whilst Taylor series, geometric series, series expansions for the exponential, sine, cosine, etc are a fundamental part of sixth form work," (Tall & Schwarzenberger 1978). Alcock and Simpson (2005) find that even when presented with a definition of convergence of an infinite series and asked to paraphrase it, directly following a unit of instruction on the topic, students differ in their descriptions of what it means for a series to converge, with a large percentage of them providing a mathematically incorrect description. Thus, it is not surprising that, as a field, we document a very wide range of ways that students think about "converge" (Monaghan 1991), in its broadest sense.

One of the most common tripping points for students in their study of infinite series comes in making the transition from studying infinite sequences to studying infinite series of numbers. While series of numbers are for mathematicians a natural extension of the study of infinite sequences, students get tangled in the complexities of the many different limiting processes that they must coordinate, as they turn the notion of infinite sequences on its side in order to reframe and accommodate their new ideas of infinite series. Namely, they must make the shift from considering simple limits of sequences to redefining an infinite series as a sequence of partial sums, and attend to the limit of *that* sequence, and not the sequence of terms itself, if they wish to make claims of convergence. Confusion abounds as students struggle with reconciling the different visual representations associated with the sequence of terms vs. sequence of partial sums, etc… and the language of limits gets used and abused as students attempt to explain the meaning of convergence in this new context. This author (in preparation) explores this idea further in a large-scale study, which identifies bottlenecks and tripping points for students, as they begin to adapt their notions of limit in these new ways.

In order to sort out students' confusion, and get a more robust understanding of what students understand about the phenomenon of convergence of infinite series – and how this is and is not consistent with their more fundamental notions of limit in calculus – it is reasonable to turn to the limit literature and ask students to explore and explain some of the tried and true examples, such as "Explain why $0.\overline{9}=1$" (from Oehrtman, 2002, for example). However, in using this task with the intention to understand what particular aspects of convergence students find salient as they make connections between limits, sequences, and series, one may have equivalently asked "Is $0.\overline{9}=1$? Please explain." or "To what does the sequence {0.9, 0.99, 0.999, 0.999, … } converge?" While mathematicians would see the consistency in these various phrasings of the same question, and reasonably assume that if students have some idea about the nature of the mathematical object $0.\overline{9}$, then they will answer consistently across all phrasings, prior literature and the study discussed in this paper indicate that this is not the case. Consider the "$0.\overline{9}$" task – Figure 1 demonstrates five different ways that this task has been used in the limit literature, dating back several decades, all for different purposes and with different results. The second column in Figure 1 displays analogous phrasings, using $\sum(1/2)^n$, henceforth referred to as the "halving" task (this author finds that students use the "halving" task more readily than $0.\overline{9}$ when asked for accessible examples of infinite series). One may guess that while students may be able to reason with particular phrasings of either of these questions, other phrasings are presented in ways that do not align with students' understanding of convergence. In the most extreme sense, these alternative phrasings may be presented in such a way that masks students' true understanding of convergence *or* brings different features of the mathematics to light in such a way that they interpret certain phrasings of the question entirely differently than others.

Some work has been done to explore the effects of multiply phrasing tasks such as these. One recent example (in a physics context), Wittmann (2012) demonstrated the differences that resulted in asking students the same task about several bulbs in a circuit by (1) framing the question as asking for both an answer *and* a justification of that answer vs. (2) framing the question by providing the answer and asking *only* for justification of that answer. In his study, Wittmann curiously found that, though many students were not able to choose and justify the correct answer for themselves, those same students were largely able to provide adequate justification for the correct answer, when it was provided.

While it may be interesting to pose the question: to which phrasings of the "halving" or $0.\overline{9}$ task do students respond correctly/consistently with a more formal understanding of the topic of convergence, this is *not* the question explored in this study. Knowing which phrasings are more or less likely to prompt 'correct' responses does not provide one with any information on the range of ideas that students associate with the topic, nor give guidance for improving student understanding. Thus, in this study, I aim to address the following research questions, which are more aligned with exploring students' extensions of their limit understanding, and more directly impact future instruction aimed at helping students grasp the difficult concept of infinite series:

- What aspects of students' understandings of convergence (of infinite series) are illuminated by the different phrasings of the "halving" question, outlined in Figure 1?
- How are these differences significant in the way we calculus educators (a) frame our teaching of this content, and (b) assess students' understanding of this content?

Data and Methods

In 2010, semi-structured interviews designed to investigate students' spontaneously generated visual representations used when explaining the topic of infinite series (of numbers) to a less-knowledgeable peer were conducted with second and third semester calculus students, and real analysis students. It became obvious during the course of the first round of interviewing that most students used some version of the $\sum(1/2)^n$ example on their own, to explain convergence, at some point during the interview. That is, it appeared to be an example with which students were relatively familiar and had some level of comfort in using to support their explanations. Thus, during the flow of each subsequent interview, the questions in the right column of Figure 1 were posed, at various times, as they became relevant and related to students' explanations and considerations through the course of their explanations. The purpose of posing these questions, as discussed above, was to explore which different aspects of convergence were illuminated with a consideration of each different phrasing of the problem.

The study discussed here makes use of a particular 1.5-hour semi-structured interview with sophomore engineering major Jenna – a representative of the larger sample of students, all of whom were participating in the related study on visual representations for infinite series. At the time of the interview, Jenna had successfully completed second semester calculus, which included an extensive unit on infinite series (taught from Stewart), with a grade of "B." Jenna was chosen as a representative case because while her responses were very typical among the larger sample of participants, she communicated them more clearly and thoughtfully than others. An in-depth, microgenetic analysis of this interaction with Jenna takes a very close look at the particular limiting processes and aspects of infinite series that were prompted by her reasoning with each phrasing of the "halving" task, and allows for a more fine-grained level of analysis than would be ascertained in other assessment situations (Calais, 2008). Such analysis with all student interviews would be impractical, so the discussion of Jenna is followed by some general patterns observed from the entire set of student interviews.

The Case of Jenna – in brief

While it is difficult to condense Jenna's work with the various phrasings of the "halving" task to such a confined space, what follows is a short description of her responses to tasks 2.1-2.5. Much more detail and discussion is provided in the more extended analysis. In the following description, of interest (as it pertains to the research questions) are the particular aspects of the "halving" task itself that Jenna considers when making conclusions about convergence. It will become apparent that in each different phrasing, Jenna shifts her attention to a different mathematical structure used in that particular task, which causes differences in the way that she views the phenomenon of convergence of this particular infinite series.

During her teaching episode, Jenna spontaneously brought up examples with both $0.\overline{9}$ and $\sum(1/2)^n$ on her own, as part of her explanation of series convergence. Her use of these examples provided entry points for all of questions 2.1-2.5 to be posed, at various relevant points in her explanation. And while the representations that she used to explain and the conclusions that she drew about the general topic of infinite series were quite consistent, Jenna showed vastly different understandings when responding to the different phrasings of the "halving" task.

In brief, when responding to task 2.1, Jenna was able to provide a complete response for why $\sum(1/2)^n = 1$. She first simply recalled the "formula" for convergent geometric series, and discussed how this fit the model of a geometric series whose common ratio of 1/2 indicated that it converged, in particular to a value of 1. When pressed by the interviewer for more reasoning and understanding, Jenna was able to provide both a "walking to the wall" metaphor (reminiscent of Zeno's Paradox - in which she used halving distances to describe the terms of the series, and the distance from her to the wall of 1 yard) and a geometric representation drawn on the whiteboard to further describe her reasoning (see Figure 2). By this work, one might claim that Jenna has some idea of what it means for an infinite series to converge, calling to attention her correct use of an appropriate metaphor, formula, and visual representation. Her examples were good, and aligned with some of the more traditional ones that might occur in a lecture on the material. She clearly communicated an idea of what a geometric series is, and was able to make conclusions based on it, beyond simply recalling the "formula" for convergent geometric series. Her work on task 2.2 extended this – in Jenna's words, "You *can* compute $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$ because even though it goes on forever … that's just a geometric series that converges because *r* is less than 1."

In her response to task 2.2, if stopping there, one might conclude that this is further evidence of her understanding the topic at hand. While it is unclear if Jenna could produce a proof or explanation of the more general case to which she refers – $\sum a(r)^n$ – it is clear that she is able to appropriately discuss the different parts of this general type of series. However, to this point in the interview, her attention has largely been on a term-by-term comparison, emphasizing the action of halving of distances or "pies" (as in Figure 2), and appears to be considering, almost exclusively, the <u>individual terms</u>, and the <u>resulting compilation of terms</u>, rather than any other limiting processes that may be appropriate in this scenario.

Continuing to reason with task 2.2, however, Jenna goes on to say "those later terms get so small that they don't matter, and after a while it doesn't change the sum." This is an inappropriate extension of limit ideas that sheds first light on some ways in which Jenna's understanding of series convergence is not correct. It also marks a shift in her reasoning pattern, away from the individual terms and toward the <u>sequence of terms</u>. That is, her attention has shifted to instead considering the ordered list of individual terms, and not just the terms independently, as she decides to consider that the later terms' magnitude is so small that it must not impact the overall sum. While consideration of independent, individual terms allowed her to justify why $\sum(1/2)^n=1$ (task 2.1), consideration of the ordered sequence of terms lead her to the same conclusion, but for a different reason, in task 2.2.

Additional contrasting evidence comes in her response to task 2.3, in which Jenna claims that, rather than producing a value of 1 (answer choice (b)), which would align with her earlier responses, the sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$ is "just less than 1, by some infinitely small value," (answer choice (c)). When probed, Jenna's unexpected answer choice was justified in a way that indicated a further shift in the mathematical structure to which she was attending. The presentation of the infinite series in task 2.3 – as an expansion with ellipses after five terms – caused her to conclude that this series must be "approaching, but not quite 1," unlike task 2.1, which was presented with sigma notation. Thus, a simple shift in the presentation of the infinite series in question caused Jenna, by her own explanation, to arrive at a different conclusion about the convergence of said series. Now, rather than attending to the individual terms or the sequence of ordered terms, Jenna attended to the <u>ellipses</u> as an indication of a different sort of limit process, for which she was uncomfortable concluding that the series converged to 1.

Further complicating matters, Jenna viewed tasks 2.4 and 2.5 as inconsistent with the other questions, and was unable to leverage her understanding to talk about partial sums or the sequence of partial sums. Task 2.5, in particular, was meant to examine whether writing the values as decimals would call something different to mind than the terms expressed as fractions. However, Jenna (as representative of many other students) was fully able to connect those values to values of "partial sums" of $\sum(1/2)^n$ without a problem. Even still, though she had previously used the language of "converges to 1," for tasks 2.4 and 2.5 Jenna said the strongest statement she could make was that it "tends to 1" or "approaches 1." In fact, when posed as a sequence of partial sums, Jenna explicitly denied that one could claim "convergence (to 1)." Jenna's work with the five phrasings of the "halving" task can be summarized (briefly) in Figure 3.

Thus, I claim that the differing responses that Jenna made to the different phrasings of the $\sum(1/2)^n$ task are interesting and useful, but that only when we consider the collection of responses to the different phrasings do we fully understand (a) the full scope of what she really intends as the meaning of "converge," (b) potential ways in which Jenna coordinates the different limit ideas with the different mathematical structures when considering infinite series, and (c) which of these understandings are dependent on task presentation vs. more deeply tied to her knowledge of the content. Thus, looking at her response to any one of these tasks can tell us about *some* aspect of her understanding, that understanding is only tied to the particular representation of $\sum(1/2)^n$ used in that task. But that is insufficient to say that she has a particular "model" (a la Williams, 1991), or "misconception" (a la Davis & Vinner, 1986; Cornu, 1991; and more). Simply asking the questions differently altered Jenna's responses, demonstrating that the different features of a particular task called to mind differences in the way that she interpreted the notion of convergence.

As will be discussed in much greater detail in the full report, though Jenna's responses were representative of the larger sample that participated in this study, there were additional patterns of responses that are significant. For example, there was a large overlap *within student*, consistent with Wittmann's (2012) findings – a significant portion of students were both able to explain why $\sum(1/2)^n = 1$ (task 2.1) while also claiming that one cannot compute $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots$ and get an answer (task 2.2). Potential explanatory factors for this discrepancy are discussed at length in the full report, but they go beyond Wittmann's suggestion that knowing the answer frees the student to justify and use reasoning, rather than trying to first make a choice and then justify it. The factors that appear to, in part, explain the overlap in this data are more closely tied to the mathematical content and the ways that students orient to infinite series as presented in a variety of different formats – with sigma notation, as a sequence of terms, as a sequence of partial sums, with ellipses v. with a limit symbol, and more. This and other patterns are identified and discussed at more length in the full report.

Contributions and Implications

Often in the present limit literature, it is common to find "results" that attempt to characterize student understanding based on their response to only one framing of, for example, the $0.\overline{9}$ task. While conclusions based on a singular phrasing of a task may be locally relevant to a particular researcher's agenda at hand, the study discussed here indicates that using a singular representation to make some claim about students' limit understanding is inadequate, as the way that the task itself is phrased has significant influence on the particular mathematical structures to which the students attend – thereby influencing the ways that they appeal to their understanding of phenomena such as convergence. From the many ways that the $0.\overline{9}$ or "halving" questions could have been posed, and the many ways that a single student could (and often does) respond differently, depending on the way it was phrased, it seems speculative to claim that any student has a particular understanding of such a concept that is not tied explicitly to the way that the mathematics was represented in the task. Thus, the findings here speak to and have implications for two general audiences. First, for those interested in how students extend their understanding of limit, it important to note how the subtleties in task presentation unintentionally bring to light differing limiting processes that have enormous impact on student interpretations of convergence. Generalizing this, broader audiences can take away not only content implications, but also methodological implications for the specificity of claims about student understanding, as they relate to task presentation.

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Figures

(1.1) Q: Explain why $0.\overline{9}=1$ (Oehrtman, 2002)	(2.1) Q: Explain why $\sum_{n=1} \frac{1}{2^n} = 1$
(1.2) Q: Can you add 0.1 + 0.01 + 0.001 + ... (the dots indicate continuation) and get an answer? (Monaghan, 2001)	(2.2) Q: Can you compute $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$ (the dots indicate continuation) and get an answer?
(1.3) Q: What is between 0.999… and 1? (a) Nothing because 0.999… = 1 (b) An infinitely small distance because 0.999… < 1	(2.3) What is the value of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...?$ (a) It does not have a value because it keeps

(c) You can't really answer because 0.999… keeps going forever and never finishes. (d) If you agree with one of the above, provide your own answer. (Szydlik, 2000)	going forever and never finishes (b) 1. (c) Just less than 1, by some infinitely small value (d) If you agree with none of the above, provide your own answer
(1.4) Q: Find the limit of the sequence: $\lim_{n\to\infty}\left(1+\frac{9}{10}+\frac{9}{100}+...+\frac{9}{10^n}\right)$ (Tall and Vinner, 1981)	(2.4) Q: Find the limit of the sequence: $\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}...+\frac{1}{2^n}\right)$
(1.5) Q: Consider the sequence {0.9, 0.99, 0.999, 0.999, … }. Which of the following is true of this sequence? (a) It tends to $0.\overline{9}$ (e) it tends to 1 (b) it approaches $0.\overline{9}$ (f) it approaches 1 (c) it converges to $0.\overline{9}$ (g) it converges to 1 (d) its limit is $0.\overline{9}$ (h) its limit is 1 (Monaghan, 1991)	(2.5) Q: Consider the sequence {0.5, 0.75, 0.875, 0.9375, 0.96875, …}. Which of the following is true of this sequence? (a) It tends to 1 (b) It approaches 1 (c) It converges to 1 (d) Its limit is 1 (e) It tends to some value that is not 1 (f) It approaches some value that is not 1 (g) It converges to some value that is not 1 (h) It has a limit that is not 1

Figure 1: Comparison of tasks – existing literature and current study

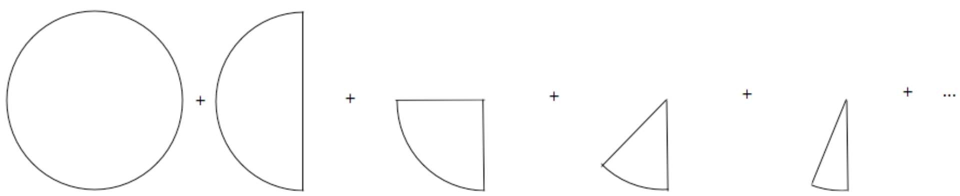

Figure 2: Jenna's geometric representation of the "halving" scenario

(2.1) Q: Explain why $\sum_{n=1}\frac{1}{2^n}=1$	Attended to: <u>individual terms</u>, independently Explanation: included geometric image (see Figure 2), "walking to wall" metaphor, and use of geometric series "formula"
(2.2) Q: Can you compute $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$ (the dots indicate continuation) and get an answer?	Attended to: <u>sequence of ordered terms</u> and the decreasing size of "eventual" terms Explanation: if the terms get "small enough" then they become "negligible" and you *can* get an answer
(2.3) What is the value of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...?$ (a) It does not have a value because it keeps	Attended to: the <u>ellipses</u> at the end of the series as an indication of continuation and uncertainty Explanation: The "dot dot dot" means "getting very close to, but not reaching"

going forever and never finishes (b) 1. (c) Just less than 1, by some infinitely small value (d) If you agree with none of the above, provide your own answer	
(2.4) Q: Find the limit of the sequence: $\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}...+\frac{1}{2^n}\right)$	Attended to: the limit symbol and the idea that there appears to be a "last term" in this version of the task Explanation: the limit is 1, but this version of the task is "not related" to $\sum(1/2)^n$ because this limit is not the same as the $\sum$ symbol
(2.5) Q: Consider the sequence {0.5, 0.75, 0.875, 0.9375, 0.96875, …}. Which of the following is true of this sequence? (a) It tends to 1 (b) It approaches 1 (c) It converges to 1 (d) Its limit is 1 (e) It tends to some value that is not 1 (f) It approaches some value that is not 1 (g) It converges to some value that is not 1 (h) It has a limit that is not 1	Attended to: the sequence as individual values that represent the various partial sums; (When prompted) shifted attention to the collection of these partial sums as an ordered sequence Explanation: the sequence "approaches 1" and "tends to 1" but does not converge to 1, because this "can never equal 1."

Figure 3: Jenna's responses to the five phrasings of the "halving" task