A Microgenetic Study of One Student's Sense Making About the Temporal Order of Delta and Epsilon

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The formal definition of a limit, or the epsilon delta definition is a critical topic in calculus for mathematics majors' development and the first chance for students to engage with formal mathematics. This report is a microgenetic study of one student understanding of the formal definition focusing on a particularly important relationship between epsilon and delta. diSessa's Knowledge in Pieces and Knowledge Analysis provide frameworks to explore in detail the structure of students' prior knowledge and their role in learning the topic. The study documents the progression of the student's claims about the dependence between delta and epsilon and explores relevant knowledge resources.

Keywords: limit, formal definition, students' prior knowledge, microgenetic study, learning

The formal definition of a limit of a function at a point, as given below, also known as the epsilon-delta definition, is an essential topic in mathematics majors' development that is introduced in calculus. We say that the limit of $f(x)$ as x approaches a is L, and write, $\lim_{x\to a} f(x) = L$ if and only if, for every number ε greater than zero, there exists a number δ greater than zero such that for all numbers x where $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. The formal definition provides the technical details for how a limit works and introduces students to the rigor of calculus. Yet research shows that thoughtful efforts at instruction at most leaves students – including intending and continuing mathematics majors – confused or with a procedural understanding about the formal definition (Cottrill et al., 1996; Oehrtman, 2008; Tall & Vinner, 1981).

Although studies have sufficiently documented that the formal definition is a roadblock for most students, little is known about how students actually attempt to make sense of the topic, or about the details of their difficulties. Most studies have not prioritized students' sense making processes and the productive role of their prior knowledge (Davis & Vinner, 1986; Przenioslo, 2004; Williams, 2001). This may explain why they reported minimal success with their instructional approaches (Davis & Vinner, 1986; Tall & Vinner, 1981). A small subset of the studies have begun exploring more specifically student understanding of the formal definition (Boester, 2008; Knapp and Oehrtman, 2005; Roh, 2009; Swinyard, 2011). They suggest that students' understanding of a crucial relationship between two quantities, epsilon and delta within the formal definition warrants further investigation. Davis and Vinner (1986) call it the *temporal order* between epsilon and delta, that is epsilon first, then delta (p. 295) and found that students often neglect its important role.

This report explores how students make sense of the formal definition of a limit in relation to their intuitive knowledge. Specifically, it investigates the micro changes in one student's understanding of the temporal order of delta and epsilon in the formal definition. Through a fine-grained knowledge analysis of student interviews, this report investigates the range of resources one student navigate through and/or refine as he developed his claim about the temporal order.

Theoretical and Analytical Framework

The Knowledge in Pieces (KiP) theoretical framework (Campbell, 2011; diSessa, 1993; Smith et al., 1993) argues that knowledge can be modeled as a system of diverse elements and complex connections. From this perspective uncovering the fine-grained structure of student knowledge is a major focus of investigation, and simply characterizing student knowledge as misconceptions is viewed as an uninformative endeavor (Smith et al, 1993). Knowledge elements are context-specific; the problem is often inappropriate generalization to another context (Smith et al, 1993). For example, "multiplication always makes a number bigger" is not a misconception that just needs to be removed from students' way of thinking. Although this assertion would be incorrect in the context of multiplying numbers less than 1, when applied in the context of multiplying numbers greater than 1, it would be correct. Paying attention to contexts, KiP considers this kind of intuitive knowledge a potentially productive resource in learning (Smith et al., 1993). This means that instead of focusing on efforts to replace misconceptions, KiP focuses on characterizing the knowledge elements and the mechanisms by which they are incorporated into, refined and/or elaborated to become a new conception (Smith et al., 1993). Documenting the micro changes in learning is one of the foci of investigation (Parnafes & diSessa, in press; Schoenfeld et al., 1993). Similarly, we view students' prior knowledge as potentially productive resources for learning. We also assume that student knowledge is comprised of diverse knowledge elements and organized in complex ways, and thus learning is seen as the process of reorganization and elaboration of students' prior knowledge.

Methods

The data is part of a larger pilot study for my dissertation where I interviewed 7 calculus students about their understanding of the formal definition. This report focuses on a case of a student, AD who used a diverse set of knowledge resources to make sense of the temporal order between delta and epsilon. AD self identified as male, and White-non Hispanic. He was an intended mathematics major, who took first-semester calculus in high school and received a 5 on his AP Calculus AB and BC. The student was selected because he "changed his mind" about the temporal order several times during the interview before arriving at the claim that epsilon came first. These changes provide opportunities for a closer look at the influence of different cued knowledge resources in his thinking and how they might have gotten elaborated or refined.

Analysis

The analysis focuses on the part of the interview where I asked the student to comment specifically about the temporal order of delta and epsilon. I broke down the part into segments based on the change in students' claim (*epsilon first, delta first* or *no order*). At times, depending on the question, the student might have characterized the temporal order in terms of dependence (epsilon depended on delta or vice versa), control (trying to control delta or epsilon), temporal order (epsilon first or delta first) or which one was set first. In ambiguous cases, I follow the position that the cued knowledge resources support. I define *knowledge resources* as relevant prior knowledge that might be used to reason and justify the issue at hand. *Cued knowledge resources* are assertions students bring up as part(s) of a mechanism to justify a currently held position or opinion. We identify cued knowledge based on what students say in the moment. Reasonable interpretations for the statement will be considered and be put through the process of *competitive argumentation* (Schoenfeld, Smith & Arcavi, 1993) using other parts of the transcripts. With each of the cued knowledge resources, particular care will be given to investigate their origin and when it originally came up, and also when possible to the students' commitment to the particular knowledge. The analysis below shows the progression of AD's claim about the temporal order, and the cued knowledge resources involved in his reasoning.

The student, AD initially argued that epsilon depended on delta (delta first) because "delta is giving you an interval for *x*, and then, like, epsilon is evaluating *x* and subtracting the limit" (turns 288-290). By the end he argued that delta depended on epsilon (epsilon first) because "epsilon's [set] first and you break down epsilon… and you find delta" (turns 393-411). Before exploring the different knowledge resources used by AD in making those claims, I report the changes that happened in the span of 17-minute episode between those two claims. The diagram below shows the changes in AD's claim about the temporal order. Each box represents a segment, and the color characterizes the overall nature of the argument about the temporal order. Red is for *delta first*. Yellow is for *no order*. Green is for *epsilon first*.

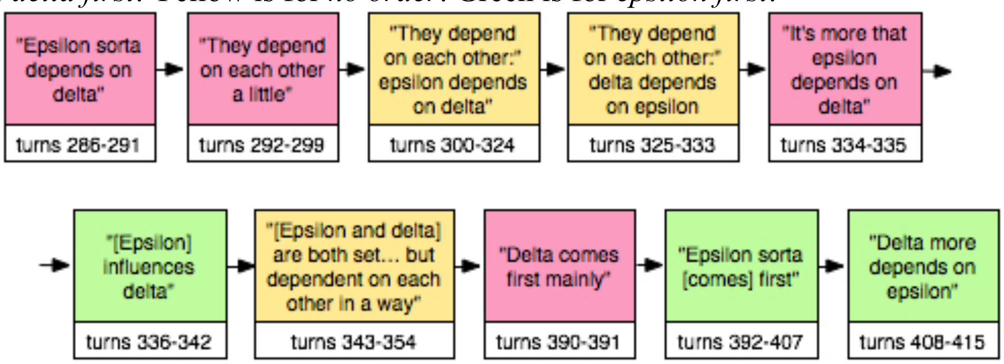

Figure 1. AD's progression of claims about the temporal order of epsilon and delta

The changes might have reflected the instability of AD's claim, but it was not due to lack of resources. Analysis shows that at each segment different knowledge resources were cued, they interacted with existing ones, and at times became conflicting. Below, I explore some of these resources in the first segment to give an idea of this process.

AD started by saying that "epsilon sort of depended on delta" in the first segment. AD argued that epsilon sort of depended on delta because delta gave an interval for *x* and epsilon evaluated the *x* and subtracted the limit from it. I claim that he cued particular views about what epsilon was and what delta *did* as well as the dependence relationship between *x* and $f(x)$.

286 AD Um delta, no, epsilon sorta depends on delta.
287 AA Epsilon depends on delta…
288 AD Because, um delta is giving you an interval for *x*,
289 AA Uh-hm.
290 AD And then like epsilon is evaluating x and subtracting the limit,
291 AA Uh-hm.

Part of AD's argument focused on what delta *did* and epsilon *was*. He said that delta gave an interval for *x* and epsilon was evaluating *x* and subtracting the limit. Earlier part of the interview indicated that AD had a more nuanced view of both delta and epsilon. About delta, AD stated that "[d]elta is another small number such that the interval, it makes the interval, sm-small, but big enough so you can actually, it's not just a point but it's uh, that you get numbers that are close to the limit" (turn 192). More than just giving an interval for *x,* delta created a particular size of interval (small but big enough), and served a particular purpose (to get numbers close to the limit). But it seems that the only aspect of delta that was cued in this turn was the fact that *delta constrained the x values*. For epsilon, his statement on 290 seems to suggest that he was saying $|f(x) - L| = \varepsilon$. Earlier accounts, however, showed that he was aware that $|f(x) - L|$ had to be less than epsilon. He stated, "Epsilon's just a number and you're using it to make sure that $f(x)$ minus L, the absolute value is just less than some certain nu- number, and it must be greater than zero, so you call it epsilon" (turns 158-160). He later also said that epsilon made sure that $f(x) - L$ was small (turns 188-190). But here, AD only cued the fact that epsilon evaluated x and subtracted the limit. Why did AD only cue certain aspects of delta and epsilon? I argue that he might have done so to cue another knowledge resource: $f(x)$ depends on x.

Focusing on the fact that delta gave him an interval for x and epsilon evaluated that x, could help establish the dependence between delta and epsilon through the dependence between x and $f(x)$. In fact, AD argued exactly this earlier, when he was explaining the meaning of the *if-then statement*, $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. He said, "… this one [*points at* $|f(x) - L| < \varepsilon$] is *dependent* on this one because in this one [*points at* $0 < |x - a| < \delta$] you're choosing the x, this [*points at* $|f(x) - L| < \varepsilon$] is evaluating the function at x" (turn 238, emphasis added). AD attended to the if-then statement to conclude the dependence, but he focused on the *evaluation* of x in order to argue for the dependence. This also suggests that AD might have also used his interpretation of the if-then statement to arrive at his conclusion. But earlier accounts showed that AD had different interpretations for the if-then statement. He treated the two inequalities in the if-then statement as two conditions to satisfy instead of an implication (turns 126-128, 142, 224, and 240). This suggests that AD used the if-then statement to infer the temporal order of delta and epsilon, but in a very particular way. He used the if-then statement to infer the purpose for delta and epsilon. That is delta sets an interval for x and epsilon evaluates such x and subtracts the limit.

In sum, in the first segment, AD cued the following knowledge resources: 1) delta constrains the interval for x; 2) epsilon is involved in evaluating the x and subtracting the limit; 3) his interpretation of the if-then statement, through which he inferred the meaning of delta and epsilon; and 4) the dependence between x and $f(x)$. The fourth knowledge resource is in fact one of the most common knowledge resources found in most students I interviewed. In the presentation, I will explore how some of these resources get *reused* and *refined* in later segments, as well as other resources that were cued during the 17-minute episode.

Conclusion and Implications

This case shows how one student made sense of the temporal order of delta and epsilon. AD's progression with the claim suggests the range of resources AD cued in the episode, and how they might have been competing resources. The analysis showed that the student cued particular aspects of his knowledge to make claims about the temporal order. It also shows that students can interpret parts of the statement of the definition in particular ways. While I was able to document the resources cued in this episode, the question remains, why do certain resources get reused and refined while others were abandoned. That is the next step in the analysis. The question for the reader would be, is the analysis demonstrated in this report convincing and informative? In what ways, can I make it more rigorous or convincing?

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