EXPERT PERFORMANCE ON ROUTINE AND NOVEL INTEGRAL APPLICATION (VOLUME) PROBLEMS

Preliminary Research Report

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Past research has shown that students struggle when applying the definite integral concept, and these difficulties stem from incomplete understanding of the integral's underlying structure. This study aims to provide insight into the construction of effective mental structures for integrals by examining experts' solutions to volume problems. Seven mathematics faculty members from a large, public university solved three volume problems (two routine, one novel) in videotaped interview sessions. Preliminary analysis shows that the experts have a rich understanding of definite integrals, and the few instances of errors seemed to be a result of inattention as opposed to a deficit in understanding. Their problem-solving process was highly structured and detailed. The experts' visual representations varied from sparse and static to fully 3-dimensional and dynamic. We hope to use this and past student data to construct a framework for analyzing student understanding of integral volume problems.

Keywords: Calculus, definite integral, volume, visualization, expert

Introduction and Literature. Determining volumes of solids is an application of the definite integral that is routinely covered in a second-semester calculus course, but very few studies have been conducted with the aim of understanding how students conceptualize these problems. Previous research has found that students have a very weak intuitive understanding of the definite integral and its underlying approximation and accumulation structure (Yeatts & Hundhausen, 1992; Grundmeier, Hansen, & Sousa, 2006; Thompson & Silverman, 2008; Sealey, 2008; Huang, 2010). In an early study on student understanding of integration (Orton, 1983), students were asked to give detailed explanations of their reasoning when solving integration problems. Orton observed that students had very little idea of the dissecting, summing, and limiting processes involved in integration. Huang (2010) observed students focusing on "calculating correctly, while ignoring the true meaning of the concepts behind the calculations."

A key component in successfully solving volume problems is visualization of the solid. Stylianou and Silver (2004) compared frequency in and nature of the use of visual representations by experts and novices while solving arithmetic word problems. They found that, even though the construction of a diagram or picture is helpful, it is the quality of the picture that is most important. Experts more frequently recognized and highlighted critical features of the visual image, which helped to focus their attention and guide the rest of the problem-solving process. For volume problems, visualization of the solid and its constituent parts guides and dictates the construction of the corresponding volume integral. Students must first translate the information given in the statement of the problem into a visual representation of the physical situation, usually by sketching. Then, they must extract specific information from their sketch and represent the information symbolically in the form of a definite integral. Successful completion of the second transfer of information (from pictorial to symbolic) requires that the student know what type of information to obtain from the sketch as well as what to do with it.

Research Question. The current work expands on the above literature in an attempt to understand how visualization ties into students' ability to set up a definite integral for tasks involving solids of revolution. The goal of this study is to more deeply explore student understanding of applications of the definite integral. The first phase involved identification and classification of common mistakes students make when setting up and solving volume problems. The second phase (discussed here) involved interviews with mathematics faculty members (which we consider to be experts) during which they solved integral volume problems of varying levels of difficulty. Our goals in interviewing experts were to observe their techniques and strategies for solving routine volume problems, and to examine how they approach novel volume problems. We are primarily interested in learning (1) if and how experts use visualization techniques like gesturing and diagrams, and (2) how each of the pieces of the underlying structure of the definite integral contribute to their problem-solving processes. Specifically for the second research question, we will compare the important elements of the structure of the definite integral (product, summation, and limit) the experts employ when solving volume problems to those employed by the subjects in Sealey's (2008) study involving other applications of the definite integral. Based on the analysis of the expert data, we aim to create a preliminary framework for analyzing student understanding of definite integral volume problems.

Theoretical Perspective. Our research is built on the foundation of the constructivist learning theory (Piaget, 1970). We certainly acknowledge that the experts in our study do not need to construct an understanding of the definite integral, but we do believe that the actions of the experts can give insight into their mental structures. The conceptual framework guiding data analysis is taken from the work done by Zazkis et. al (1996) on student understanding of the dihedral group D_4. The Visualization/Analysis (VA) model deals with both visual and analytic thinking and how these two types of cognition are used together in problem solving. Application problems in calculus almost always require a transfer of information from one form to another (written, pictorial, symbolic, numerical, etc.) and volume problems in particular involve the transfer of information from visual representation into symbolic form. Using the VA model, we will be able to more systematically examine and categorize understanding of integral application problems by focusing on the instances of information transfer during the problem-solving process.

Research Methodology. Interviews with experts (mathematics faculty members) were conducted during the Fall 2012 semester at a large, public, research university. The seven participants were faculty members of a mathematics department who responded to a faculty-wide e-mail sent out by the primary investigator. The sample included four full professors, two assistant professors, and one teaching assistant professor. Two of the professors are female and five are male. The sample also varied in how recently the professors had taught a course that included discussion of integral volume problems – four professors had taught integral volume problems within the past five years, and three professors had either never taught the topic or had taught it more than five years ago. While we acknowledge that their ability to recall information about volume problems may be weak, we still consider all of the participants to be experts in mathematics.

During the interviews, the participants were asked to complete three problems concerning volumes of solids of revolution; the first two problems were considered "easy" or routine problems that are typically solved by students in a second semester calculus course, and the third was more complex than would be expected in such a course (see Appendix A). As they worked through each problem, the participants were asked to think out loud and elaborate on what they had written. The experts were asked to first approach each problem from an expert point-of-view and then discuss how they would teach the concept in class. Each interview was videotaped and transcribed for analysis.

Preliminary Results. In the first phase of this study, students exhibited errors in nearly every component of the problem: variable of integration, bounds of integration, integrand formula, accuracy of sketch, and relationship between sketch and integral set-up. The experts' mistakes coincided with students' mistakes, although the experts' mistakes were less numerous, and no difficulties with sketching or determining the variable of integration were observed.

As expected, the experts had very few difficulties with the first two problems, although only two of the seven were able to complete both routine problems error-free. More errors occurred in the first problem than the second problem despite the fact that the second problem is generally considered to be more difficult. We suspect that if we had switched the order of the problems, the experts would have more errors in whichever problem they attempted first, and we do not see the errors as being a conceptual misunderstanding with these experts, but instead an inattention to details. Interestingly, a mistake that would be considered evidence of a student's complete misunderstanding of integral volume problems was made by two different experts on the first problem. Experts' visual representations varied greatly with respect to aspects like level of detail, use of dynamic indicators like rotation arrows, and 3-dimensional details.

The third problem was indeed novel to all seven participants despite the fact that second-semester calculus knowledge is sufficient to solve it. Our aim in presenting the experts with a novel problem from a relatively simple area of mathematics was to get them out of their comfort zones and to see how their methods for solving the routine and novel problems compared. Even though a majority of the experts approached the third problem by first rotating the coordinate axes – a technique that may not be accessible to most second-semester calculus students – we believe that analysis of this data has the potential to produce useful insight into effective thinking. At this time, it seems as though the ways in which the experts attended to and explained the "*dx*" in the first two problems are indicators of how their solution strategies and performance on the novel problem. As of the submission of this proposal, data analysis is still in the preliminary stages, so we look forward to being able to share more of our findings at the conference.

Future Research/Teaching Implications. The research discussed here is one phase of a larger study on student understanding of the definite integral when applied to finding volumes of solids. We plan to use the expert interview findings in conjunction with previous research on integration to produce a framework for analyzing student understanding of the definite integral. Volume problems can be considered as somewhat atypical compared to most single-variable calculus application problems due to their highly visual nature. The vast majority of physical situations encountered in first- and second-semester calculus application problems are 2-dimensional, and there are many physical situations discussed that cannot be accurately represented by a picture (e.g., work). We believe that this aspect of volume problems makes them a powerful tool in improving students' mathematical maturity and strengthening their ability to transfer knowledge between different representational domains.

Questions for Audience.
1. How does analysis of expert problem-solving techniques contribute in a meaningful way to advancing student understanding?
2. Are there any other aspects of expert data to which we should be attending?
3. How do we learn from our experts' mistakes?
4. Are there any examples of "real-world" problems or situations in which integration is used in determining the volume of a 3-dimensional object (e.g., geology)?

Appendix A. *Interview Problems*

For each problem, set up the integral that gives the volume of the solid.

1. The solid obtained by rotating the region bounded by the curves $y = x^3$, $y = 8$, and $x = 0$ about the *y*-axis.

2. The solid obtained by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{x}{2}$ about the line $y = 3$.

3. The solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = x$ about the line $y = x$.

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