MATHEMATICIANS' EXAMPLE-RELATED ACTIVITY WHEN PROVING CONJECTURES

Elise Lockwood, Amy B. Ellis, & Eric Knuth
University of Wisconsin – Madison

Examples play a critical role in mathematical practice, particularly in the exploration of conjectures and in the subsequent development of proofs. Although proof has been an object of extensive study, the role that examples play in the process of exploring and proving conjectures has not received the same attention. In this paper, results are presented from interviews conducted with six mathematicians. In these interviews, the mathematicians explored and attempted to prove several mathematical conjectures and also reflected on their use of examples in their own mathematical practice. Their responses served to refine a framework for example-related activity and shed light on the ways that examples arise in mathematicians' work. Illustrative excerpts from the interviews are shared, and five themes that emerged from the interviews are presented. Educational implications of the results are also discussed.

Keywords: Examples, Proof, Mathematicians

Introduction

Much of the current literature on teaching proof in school mathematics underscores the goal of helping students understand the limits of example-based reasoning (e.g., Harel & Sowder, 1998; Stylianides & Stylianides, 2009; Zaslavsky, Nickerson, Stylianides, Kidron, & Winicki, 2012) and typically characterizes example-based reasoning strategies as obstacles to overcome. However, given the essential role examples play in mathematicians' exploration of conjectures and subsequent proof attempts, example-based reasoning strategies should not be positioned only as barriers to negotiate. Indeed, the field may benefit from a greater understanding of the ways in which those who are adept at proof, such as mathematicians, critically analyze and leverage examples in order to support their proof-related thinking and activity. While the role of examples in learning mathematics more generally has received attention in the literature (cf., Bills & Watson, 2008), considerably less attention has been directed toward the specific roles examples play in exploring and proving conjectures. In this paper, we examine mathematicians' thinking as they explore and develop proofs of several conjectures. We report themes that arose during the interviews and discuss potential implications for the teaching and learning of proof.

Literature Review and Theoretical Framework

Epstein and Levy (1995) contend that "Most mathematicians spend a lot of time thinking about and analyzing particular examples," and they go on to note that "It is probably the case that most significant advances in mathematics have arisen from experimentation with examples" (p. 6). Clearly, examples play a critical role in mathematicians' development of and exploration of conjectures, and there is often a complex interplay between mathematicians' example-based reasoning activities and their deductive reasoning activities (e.g., Alcock & Inglis, 2008). Several mathematics education researchers have accordingly examined various aspects of the relationship between example-based reasoning activities and deductive reasoning activities among both mathematicians and mathematics students (e.g., Antonini, 2006; Buchbinder & Zaslavsky, 2009; Iannone, Inglis, Mejia-Ramos, Simpson, & Weber, 2011; Knuth, Choppin, &

Bieda, 2009;). The study presented in this paper builds directly upon a framework developed by Lockwood et al. (2012) that categorizes types of examples, uses of examples, and example-related strategies reported by mathematicians in a large-scale open-ended survey. Due to space, only part of the framework (Uses and Types of examples) is presented in Figures 1 and 2 below. This framework guided the coding of the interviews in this study and served to situate the themes presented below.

<Insert Figure 1> and <Insert Figure 2>

Methods

The data presented in this paper come from interviews that were conducted with six mathematicians as they explored and attempted to prove several mathematical conjectures (see Figure 3). Five of the mathematicians have a doctorate in mathematics, and one has a doctorate in mathematics education; all are currently faculty in university mathematics departments. All of the mathematicians were given Conjectures 1 and 2, and three each did Conjectures 3 and 4, which were randomly assigned. After working on each conjecture, the mathematicians were asked clarifying questions about their work. In addition, at the end of the interview they were asked reflective questions about their example-related activity, both that they had done during the interview, and more generally in their personal work. They were given approximately 15-20 minutes to explore each conjecture; although typically they were not able to complete proofs for each of the conjectures in the time allotted, they were able to make progress toward that end. (Note, that our interest was in their example-related activity while exploring and attempting to develop proofs, not in the proofs they may have produced given more time.)

<Insert Figure 3>

Conjectures 1-3 were taken from Putnam Exams, and Conjecture 4 was adapted from tasks in Alcock & Inglis (2008). We chose these problems for two primary reasons. First, the conjectures were accessible to the mathematicians (regardless of their area of expertise), but were not so clearly obvious that they could be proven immediately. Second, the conjectures were also accessible to the interviewer, allowing her to follow the mathematicians' work as well as to ask meaningful follow-up questions. While the choice of such conjectures may result in something of an inauthentic situation for the mathematicians, the choice did enable us to observe what mathematicians do as they actually explore and attempt to prove conjectures.

The interviews were transcribed, and a member of the research team analyzed them using the aforementioned framework (Lockwood et al., 2012). The process involved coding both mathematicians' observable example-related activity and their reflections about examples. The entire research group reviewed data excerpts that were difficult to code. These codes served to refine the initial framework, and the organizing of the codes in turn resulted in a number of themes about mathematicians' example-related activity in exploring and proving conjectures (Strauss & Corbin, 1998). We present these themes as the major results of this paper, as they shed light on how people who are adept at proof interact with examples as they consider conjectures.

Results

In this section, we share five main themes that arose from our analysis of the interview data. These themes not only illuminate the role examples play in the proving process for mathematicians, but also suggest implications regarding the role examples might play in classroom settings. Given the proposal's page limits, we do not go into great detail about these results; however, we do provide representative interview excerpts to illustrate the themes.

Theme 1 – There is a back and forth interaction between proving and disproving

All six mathematicians discussed the role of counterexamples in their proving process, noting that as they attempt to develop a proof, they engage in a back and forth process of formulating a proof and considering counterexamples. They described starting out by attempting to prove a conjecture, but then may get stuck, stop, and search for a counterexample. This search for (or inability to find) a counterexample might then provide insight into the development of the proof. An example of this is seen in Mathematician 2's reflection about his work with examples.

M2: You're trying to prove something and you go ahead and you try to prove it. And you realize that you're stuck at some point…Here's this gap. I start saying let's try, out of that gap, to build a counter example…Then you spend some time trying to build that object. And if you can't, then you try to sort of distill why can't you? And do the reasons why you can't build that, does that now fill in the gap in your proof? If it does, great. You've now pushed your proof further or maybe you've completed the proof entirely. And if it doesn't, then it refines what...the counter-example would have to look like…And so it's this sort of back and forth trying to use that. You know build a counter example and the failure or success of that to go back and look at what that says about your proof. And that dynamic back and forth can sometimes bear some fruit.

Theme 2 – Context and familiarity directly influence example choice

Four of the mathematicians also noted that context and familiarity have a direct impact on their selection of examples, often enabling them to make well-informed choices. Specifically, mathematicians indicated that if they were working in domain they knew well, they would regularly draw upon familiar, or "stock," examples. For instance, on the deficient number problem (Conjectures 4a and 4b), Mathematicians 4 and 6 clearly used their familiarity with the fact that 6 is a perfect number to make progress on that task, as seen in Mathematician 4's exchange below.

M4: Conjecture 4a: A number is abundant if and only if it is a multiple of six. Hmmm ok. So an example immediately comes to mind. Six is a perfect number and so that's going to be false if you are allowed to take a trivial multiple of six. So….
I: …Ok. And that you knew six was a perfect number from experience.
M4: Yeah, that one I just happen to know.

In other cases, if the domain was less familiar, the mathematician might rely on examples to make sense of the conjecture. This is exemplified in Mathematician 6's reflection below. Here we see that in a familiar domain he might simply launch into proving without having to consider examples, but that when he is "completely clueless" he tries to generate examples.

I: … Can you describe the role of examples in your work with mathematical, mathematical conjectures? How do you choose then? Do you have strategies for example-related activity? Like if you were to have to reflect on how you would use examples?
M6: So, well, first of all, it depends on the, the domain. I mean, there's some domains when I know, very familiar with all of the, like the more algebraic, formal techniques…and I can kind of recognize if it's a situation where I can actually get by without even really understanding…the problem, because I can just throw the tools at it…and it'll fall out… Other than that, I usually try to, I go in a couple different ways. Especially if I'm completely

clueless about what's going on, then I will usually use an example to try to figure out what's the conjecture is saying.

Theme 3 – Examples can lead to proof insights, both into whether the conjecture is true or false, and into how a proof might be developed

There were two ways in which mathematicians seemed to use examples to gain some insight into their proving process. First, examples served to inform whether or not a given conjecture might be true or false. At some point each mathematician used an example to decide whether he should go about trying to prove or disprove the conjecture. Second, examples served a richer purpose than simply shedding light on whether a statement was true or false. On several occasions mathematicians used specific features of an example in order to make significant steps toward a proof. In these instances, the mathematicians seemed to ground their thinking in a particular example, and by manipulating that example they developed an idea for how a more general proof might develop. As an example of this, we highlight Mathematician 6's work on Conjecture 4b as he tried to prove the contrapositive of the statement (that a number with factors that are not deficient must itself not be deficient). Mathematician 6 was examining what he called "test cases," in which he drew upon the perfectness of 6 to examine numbers in which 6 was a factor. His rationale for this is seen below.

M6: And then the real reason why I went after it with examples, not so much that I thought these would be counterexamples, as I thought they would be good test cases. And they'd maybe give me a feel for how, more information as to maybe why this is true.
I: Okay, and what do you mean by test case?
M6: Um, test case because the six, like I said before is perfect. So it's going to be, it's a, it's a pretty decent, uh, example of maybe, it's, so if anything has a chance to be a divisor that's not deficient inside of number that is deficient… I would guess it would be a perfect number.

Continuing to focus on 6, after trying to see if 6*2 and 6*3 would have to be abundant, he chose an example of 6*11. While working through this example, he had the following insight:

M6: It's almost like you get, like a duplication of the perfect-ness of six that shows up in this piece here.
I: Okay, how so?
M6: So, so, like this one, two, three adds up to six. Eleven, twenty-two, thirty-three actually adds up to sixty-six. So I'm feeling like I probably ought to be able to prove that this is a true statement.

His work with this example not only confirmed that he thought he could prove the conjecture, but work with the 6*11 example led him to make particular observations about the problem (in this case, the specific way in which certain factors added up). These observations pointed him to a more generic argument, and ultimately led him to a correct sketch of a proof.

Theme 4 – Knowledge of mathematical properties inform example choice

Another feature of the role of examples was that the mathematicians capitalized on their understanding of mathematical properties as they selected their examples. They took into account the domain to which the conjecture pertained (such as number theory or algebra), and they used that knowledge to pinpoint examples with certain properties. Their mathematical expertise came through as they spoke about mathematical features of their examples, such as choosing a number that is highly divisible or creating a set with no primes. This emphasis on properties came out most frequently with Conjecture 1, as the mathematicians tried to consider examples or counterexamples of the conjecture. In this case, the mathematicians clearly drew upon their knowledge of mathematical topics such as primeness, common divisors, the fundamental theorem of arithmetic, etc. As an example, Mathematician 3 constructed a set {4, 8, 12, 20} in an attempt to derive a counterexample. He had recognized that a counterexample must not have primes in it, and the excerpt below highlights his consideration of specific mathematical properties as he attempted to construct a possible counterexample and proceed with the problem.

> M3: The greatest common divisor between the two of them [looking at the statement of the conclusion] is not prime…Okay, it would have to be some set like 4 [writes {4, 8, 12, 20}]. That would be…their greatest common divisor is not prime. But, for every integer, the question is… There are some *n*s where the greatest common divisor is one or S…okay this one seems true, because if *n*, there are going to be integers which are multiples of S…But if the other ones here all have in common more than a prime [referring to the four numbers in his set]… So, so, if this were not true, that would mean that every two of these [referring to the four numbers in his set] have a composite number as a greatest common divisor.

In considering what might be needed to make a counterexample, Mathematician 3 displays knowledge of elementary number theory as he carefully selects four numbers that are not prime and that all have a composite number as a greatest common divisor. Facility with specific mathematical properties enabled him to make sophisticated decisions in constructing an example.

Theme 5 – Multiple examples can lead to meaningful patterns, resulting in conjecture generation and proof development

Five of the mathematicians demonstrated an explicit awareness of the relationship between examples and patterns in their work. As they worked through the conjectures, some mathematicians tried a series of examples that suggested they sought a pattern that could help them develop a proof. In his reflection on his own mathematical research, Mathematician 3 said that "he wouldn't come up with a conjecture without some examples" and suggested that looking for patterns through examples was the very activity that often led to conjectures. Mathematician 4 similarly noted that typically his work with conjectures is not externally motivated (such as solving the interview tasks), but that in his own work, examples that form a pattern tend to be the motivation for the conjectures that he formulates and ultimately wants to prove. Such statements from the mathematicians provide insights about how finding patterns in examples can ultimately lead to formulating, and perhaps eventually proving, conjectures.

Conclusion and Implications

Although the results presented here are based on a small set of interviews with mathematicians, the results are consistent with the results from our large-scale survey of mathematicians and their responses about their work with examples (Lockwood et al., 2012). The interview data highlight that examples play an important and meaningful role in the proof-related activity of mathematicians. Clearly, mathematicians possess an awareness of the powerful role examples can play in exploring, understanding, and proving conjectures, as well as the ability to implement example-related activity in meaningful ways. Yet, the role examples play in proof-related activities in mathematics classrooms, secondary school classrooms as well as undergraduate classrooms, often stands in stark contrast to the role examples play in the proof-related activities of mathematicians. Such a contrast between the role examples play in the work of mathematicians and in the work of students highlights the need for explicit instruction on how to strategically think about and analyze examples in exploring and proving conjectures—instruction students rarely, if ever, receive. Indeed, if students are to develop such awareness and ability, it is important to help them learn to think critically about how they can draw upon examples as they engage in exploring and proving conjectures.

Example Type	Definition
Simplicity	Expert appeals to an easy, simple or basic example. Includes "trivial" and "small."
Counterexample /Conjecture Breaking	Expert picks an example that might disprove the conjecture. The expert might explicitly say "a counterexample," but this can also be inferred.
Complex	Expert picks a complex example in order to test whether the conjecture holds for tricky ones; synonyms include "non-nice," "non-trivial," or "interesting."
Easy to Compute	Expert chooses an example that is easy to manipulate. The difference between this code and "Simple" is that the expert says something about working the example out.
Properties	Expert takes into account some specific mathematical property – he or she might reference a "property" or "features," or might mention particular properties.
General/Generic	Expert uses general or generic examples, or describes examples that are seen as representative of a general class of cases or otherwise lack special properties.
Boundary Case	Expert picks an extreme example or number, or a "special" case, such as the identity.
Familiar/Known case	Expert chooses an example with which he or she is familiar, or in which properties related to the conjecture are already known.
Unusual Examples	Expert picks an unusual number, which would be described as something that does not come up often. "Rare," "obscure," "strange," and "weird" are also synonyms.
Random	Expert describes the example as randomly chosen; this includes mathematical randomness, such examples chosen with a random number generator.
Exhaustive	Expert looks for "all" of the examples in an exhaustive manner. This can be by testing all possible examples or by using a computer.
Common	Expert describes the example as typical, common, or one many would choose.
Dissimilar Set	Expert indicates that he or she purposely selects a variety of types of examples.

Figure 1 – Types of Examples

Example Use	Definition
Check	Expert selects examples to make a judgment about the correctness of a conjecture; "test," "verify," and "check" are all synonyms.
Break the Conjecture	Expert tries examples to break the conjecture; this can include specifically looking for a counterexample.
Make Sense of the Situation	Expert uses an example to deepen his or her understanding of why the conjecture might be true or false, or to gain mathematical insight.
Proof Insight	Expert indicates that his or her production of examples (or counterexamples) might have a direct bearing on understanding how to prove the conjecture.
Generalize	Expert mentions using the example to generalize or to allow the expert to work in a more general situation.
Understand Statement of the Conjecture	Expert uses an example to better understand the statement of the conjecture.

Figure 2 – Uses of Examples

Conjecture 1
Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Prove that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

Conjecture 2
Let n be an even positive integer. Write the numbers 1, 2, …, n^2 in the squares of an n x n grid so that the kth row, from left to right, is $(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n$. Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black.
Prove or disprove: For each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Conjecture 3
Let S denote the set of rational numbers different from {-1, 0, 1}. Define $f : S \to S$ by $f(x) = x - 1/x$.
Prove or disprove: $\bigcap_{n=1}^{\infty} f^{(n)}(S) = \varnothing$, where $f^{(n)}$ denotes f composed with itself n times.

Conjecture 4
All the numbers below should be assumed to be positive integers.
Definition. An abundant number is an integer n whose divisors add up to more than $2n$.
Definition. A perfect number is an integer n whose divisors add up to exactly $2n$.
Definition. A deficient number is an integer n whose divisors add up to less than $2n$.

Conjecture 4a. A number is abundant if and only if it is a multiple of 6.
Conjecture 4b. If n is deficient, then every divisor of n is deficient.

Figure 3 – The conjectures given to the mathematicians

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