ODD DIALOGUES ON ODD AND EVEN FUNCTIONS

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A group of prospective mathematics teachers was asked to imagine a conversation with a student centered on a particular proof regarding the derivative of even functions and produce a script of this imagined dialogue. These scripts provided insights into the script-writers' mathematical knowledge, as well as insights into what they perceive as potential difficulties for their students, and by extension difficulties they may have had themselves when learning the concepts. The paper focuses on the script-writers' understandings of derivative and of even/odd functions.

Key words: [Odd/even Functions, Proof-scripts, Calculus, Derivative]

There is an old adage that in order to truly understand something you have to teach it. In accord with this idea, the activity of teaching and thinking about teaching can reveal a lot about the teachers' own conceptions of the subject matter. In this study, a group of prospective secondary school mathematics teachers was invited to imagine their interaction with students about a given theorem and its proof, and describe it in a form of a script for a dialogue between a teacher and a student, referred to as a *proof-script*. Through writing a script the participants provided a glimpse into their own mathematical knowledge as well as into what they perceive as a potential difficulty for their students. In particular, I focus on participants' understanding of the concepts of derivative and of even/odd functions, as featured in their composed scripts.

Background

The research reported in this article concerns the theorem "The derivative of an even function is odd", and a particular proof of this result, which is accompanied by the diagram presented in Figure 1. This was studied using a proof-script methodology. As such, brief notes on scripting and on the concepts that appear in the theorem are provided below.

Scripting

The idea of scripting was inspired by the "lesson play" construct (Zazkis, Liljedahl & Sinclair, 2009, Zazkis, Sinclair & Liljedahl, 2013). Lesson play, as a method of preparing for instruction, presents a lesson or part of a lesson as a script of interaction between a teacher and students. Zazkis, Sinclair & Liljedahl reported on the use of this tool with prospective elementary school teachers. However, they suggested that the tool could be easily extended and adopted in other contexts. This report presents one such extension.

On Odd and Even Functions

There are several equivalent formulations of the definitions of odd functions and even functions. Analytically, even functions satisfy the property that $f(x)=f(-x)$. They can be defined graphically as functions that have a reflectional symmetry about the y-axis. Similarly, odd functions can be defined graphically as functions which have a 180° rotational symmetry about the origin, or analytically as functions for which $f(x)=-f(-x)$.

The reason odd and even functions are given the same names as subsets of the integers is tied to the properties of monomials of the form x^n where n is an odd/even number. But this does not tell the whole story, since the definition of odd and even functions are not restricted to monomials; it applies to all of functions that have the appropriate symmetry properties. In the case of polynomials any polynomial with only even exponents is an even function (e.g., $f(x)=3x^6-4x^2+x^0-x^{-4}$) and any polynomial with only odd exponents is an odd function (e.g., $g(x)= 3x^7+4x-x^{-3}$). This is also related to McLaurin series expansions. McLaurin series expansions of even and odd functions consist of only even and odd powers, respectively (Sinitsky, Zazkis, & Leikin, 2011).

Method

Fourteen prospective secondary school mathematics teachers participated in the study. They held degrees in either mathematics or science, and as such, all had several Calculus courses in their formal mathematics background. They were enrolled in their final semester of a teacher education program. The participants were asked to respond in writing to the Task presented in Figure 1.

The Task sates a theorem, "The derivative of an even function is an odd function", and presents a proof for this theorem accompanied by a diagram. Both the proof and its accompanying diagram are borrowed from Raman (2001), who identified it as an intuitive but non-rigorous proof. The participants were invited to write a script for a dialogue with an imaginary student (or a group of students) in which the teacher examines the students' understanding of the proof and explains, as necessary, issues that are potentially problematic.

I was interested in exploring the difficulties that perspective secondary teachers envision their students might have in understanding the given proof. Additionally, I explored what might be revealed by the script-writing method about participants' personal understanding related to the concepts of derivative and odd/even functions.

Theoretical Framework

Hazzan's (1999) reducing abstraction framework serves as a theoretical lens for analyzing the data. This framework is based on three different interpretations of *levels of abstraction* discussed in the literature: (a) Abstraction level as the quality of the relationships between the object of thought and the thinking person, based on Wilensky's (1991) assertion abstraction is a property of a person's relationship with an object; (b) Abstraction level as a reflection of process-object duality, based on Sfard's (1991) distinction between process and object conceptions of mathematical objects; and (c) Abstraction level as the degree of complexity of the concept of thought, based off the assumption that the more compound an entity is, the more abstract it is. These interpretations function as modes through which students can reduce the abstraction level of a mathematical object. It is important to note that they are neither mutually exclusive nor exhaustive. In terms of Hazzan's framework that means that it may not always be possible to categorize every instance of reducing abstraction with exactly one of the above interpretations.

Results and Analysis

My analysis of the composed scripts focuses on the mathematical ideas that appear in the scripts, as related to even and odd functions and to the concept of derivative. These are attributed to either a teacher-character or a student-character. Throughout this analysis I use the terms 'participants' and 'scrip-writers' interchangeably to describe the authors of the scripts and use the terms 'student' and 'teacher' exclusively to refer to the fictional characters created by the scripts' authors.

On the Meaning of 'Odd'

The script-writers demonstrated an awareness of the fact that the terms 'odd' and 'even,' when attributed to a function, may cause some initial difficulty for students who try to connect these terms to their understanding of odd and even numbers. The following excerpts exemplify such awareness:

Excerpt 1 (Gina)

Teacher: Alright, so now that we have a good understanding of what an even function is does anyone have any ideas about odd functions?

Student: Are they functions that are not symmetrical about the y-axis?

Excerpt 2 (Carl)

Student: OK, well we know that an even function reflects over the y-axis and I know that even and odd are just opposites. So maybe that means that the function is reflected over the x-axis?

Excerpts 1 and 2 take place after the idea that even functions are symmetric about the y-axis has already been reiterated. The student character's conjecture that odd functions are "not symmetrical about the y-axis" (excerpt 1) is rooted in his knowledge of numbers, where an odd number is a number that is not even. This is explicitly mentioned in excerpt 2, where a student refers to even and odd as "just opposites". 'Opposite' is interpreted as a reflection over the x-axis.

This initial interpretation by students can be seen as a way of reducing abstraction, where a new and unfamiliar term, odd function, is thought of in terms of a familiar term, odd number. The teachers in these excerpts went on to correct the "symmetric about the x-axis" misconceptions and thus the misconception cannot be attributed to the script-writers. However, it is an indication that those participants deemed the terms odd/even when used to describe function to be misleading and likely to cause confusion with the number theoretic meaning of these terms.

On Examples and Definitions

Consider the following excerpts from the scripts.

Excerpt 3A (Gail)

Teacher: Well let's look at some examples. One of the examples of an even function that you gave was $f(x) = x^2$. What is the derivative of x^2?"

Excerpt 4A (Mike)

Teacher: Let's start calling the functions with the even polynomials EVEN functions and the odd polynomials ODD functions.

The common feature of these excerpts is their focus on a monomial function of the form $f(x) = x^n$. In excerpt 3A, $f(x) = x^2$ is treated as a particular example of an even function. However, in excerpt 4A, the monomial-based definition is introduced as an implicit agreement between the characters. From the discussed examples it is clear that the vaguely used term "even polynomials" is related to the parity of the exponent in $f(x) = x^n$.

Only two dialogues mentioned examples of an even function other than $f(x) = x^n$, for an even n: these were $f(x) = \cos(x)$ and $f(x) = |x|$. Others relied solely on $f(x) = x^n$, either as an example or as an informal definition of even and odd function, based on the parity of n. In other words, the invoked example space (Watson & Mason, 2005) for even and odd functions for the majority of participants was limited to monomial functions of the form $f(x) = x^n$. It may be the case that the diagram reminded participants of a parabola of the form $f(x) = x^2$, as is discussed in a next section.

Limiting the example space to monomial functions creates a convenient relationship in connecting the parity of the function to the parity of its derivative. Consider how the scripts in the above examples continue:

Excerpt 3B (Gail)

Teacher: Now can you tell me the rule you used to determine the derivative of x^2?

Student: Well you bring the exponent down in front of the x and then the new exponent becomes the old one minus one.

Teacher: We can generalize that in a formula $\frac{dy}{dx}x^n = nx^{(n-1)}$

If you were using this general rule, can you see why the derivative of an even function is always odd?

Student: I see, if n is an even number, the function is even. Then if you take the derivative n-1 will be an odd number so the derivative is an odd function.

Excerpt 4B (Mike)

Student: If I were to take the derivative of $f(x) = x^2$, then I would get $g(x) = 2x$; where g(x) is the derivative of f(x). […] This makes sense for all functions because you reduce your polynomial by a factor of one when you take the derivative, so any even number should be subtracted by one and will yield an odd number.

These excerpts explicitly mention how to find a derivative of polynomial functions and explicitly limit the example space to polynomials of a particular form. They also assume that function notation uses only a single exponent.

The script-writer's focus on a particular example of a mathematical notion rather than that notion itself is one of the ways of reducing abstraction. To reiterate, reducing abstraction is a mechanism of making sense by considering mathematics on a less abstract level than is suggested by an instructor or a text (Hazzan, 1999). Here the abstract concept of an even/odd function is replaced by a particular cluster of examples of such functions, functions of the form $f(x) = x^n$. Further, consideration of monomial functions can also be interpreted as an instance of reducing abstraction by focusing on familiar, or at least more familiar functions, both in terms of their graphs and in terms of the rules of finding a derivative. As mentioned, various forms of reducing abstraction are not mutually exclusive.

On The Use of a Diagram

The proof of the theorem provided in the Task for participants was accompanied by a diagram. However, while the diagram provides a helpful reference point, complete reliance on a diagram may lead to possible misconceptions, as exemplified by the following excerpts.

Excerpt 5 (Olga)

Teacher: Yes. Now looking at the diagram again, what sign will the slope have for f(-x)?

Student: f(-x) will have a negative slope.

Excerpt 6 (Beth)

Teacher: Great Jeff. Now I want you to look at the picture here. On the right side of the graph, is the slope positive or negative?

Jeff: Positive.

Teacher: Yes, can anyone tell me why? Ah… Laura.

Laura: Well both x_2 and y_2 are bigger that x_1 and y_1, so we will have a positive/positive, leaving using us with a positive slope.

Teacher: Perfect, now let's look at the slope at the point –x. Will the slope be positive or negative?

Laura: Negative.

Excerpts 5 and 6 explicitly reference the diagram, and identify the slope at (-x) as negative. This fails to capture the idea that the slopes get opposite values, regardless of which one is positive. Referring to the positive slope on the right side of the graph demonstrates a reliance on a diagram, which is not generalized. While the diagram assists in understanding the proof, it also limits the example space of functions that satisfy the condition of evenness. Reliance, or in this case over-reliance, on a diagram can be seen as yet another method of reducing abstraction. The diagram is seen as representing a class of functions that decrease for negative values of x and increase for the positive ones. As such, a particular example, or a class of examples, rather than a general case is being considered, which is in accord with interpretation (c) of the reducing abstraction framework.

Summary and Conclusions

The proof-script method of studying students' mathematical understanding evolved out of work on lesson plays (Zazkis et al., 2009, 2013). This paper contributes to an understanding of the utility of this method and adds to the literature regarding students' understandings of and approaches to odd/even functions and derivative.

In general, the script-writers showed evidence of being comfortable with both analytic and graphical modes of representing odd and even functions, drawing appropriate connections between the two. In several scripts student-characters showed initial confusion regarding the terms odd/even as related to functions, indicating that the script-writers were aware of the incongruence between the usage of these words in function context and in number theoretic contexts. However, the participants' invoked example spaces (Watson & Mason, 2005) triggered by the task of explaining the theorem seemed fairly limited. Most mentioned only monomials of the form $f(x)=x^n$ and only two of the 14 participants mentioned odd/even functions that were not polynomials. This also influenced the discussion of derivative. Most scripts noted the relationship of derivative and slope, however, the treatment of derivative was limited to that of monomial functions.

In several scripts it was mentioned that even functions are decreasing for negative values of x and increasing for positive values. I believe this is due to a combination of over-reliance on the provided diagram with limited sample space. How deeply rooted and systemic this issue is a topic for further research. It may be the case that some participants would be resistant to identifying even graphs which are increasing for negative values of x and increasing for positive values of x (e.g. $f(x)= -x^2$) as even functions.

Instances of reducing abstraction were evident in all of the proof-scripts. These instances arose with both student and teacher-characters. When attributed to student characters these can be interpreted as the script-writers' awareness of the difficulties students may face in learning the concepts and strategies those students may use to cope with these difficulties. These may be influenced by artifacts left from script-writers' own learning experiences. However, the instances of reducing abstraction either introduced or reinforced by teacher-characters can be seen as indicative the script-writers' own approaches and strategies. The concretization of odd/even achieved through limiting example space to the set of monomials was the most ubiquitous example of this tendency.

The method of engaging participants' in script-writing appeared useful in investigating their understanding of mathematical concepts. While the scripts also provide interesting information about the envisioned pedagogical approaches, these were mentioned only in passing and could serve as a focus of future research.

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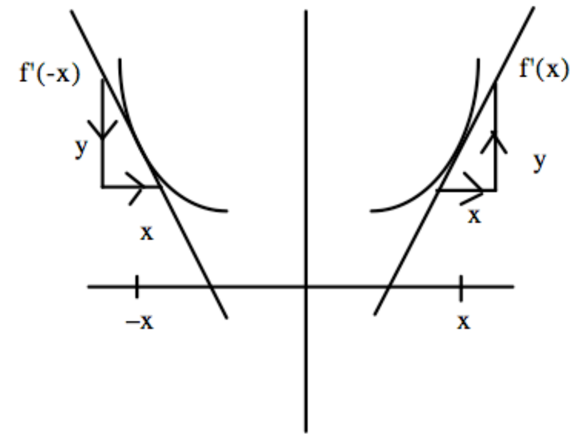
Theorem: The derivative of an even function is an odd function.

Consider the following proof of this theorem:

If f(x) is an even function it is symmetric over the y-axis.

So the slope at any point x is the opposite of the slope at (-x)

In other words, f'(-x) = - f'(x) , which means the derivative of the function is odd.



Imagine that you are working with a student and testing his/her understanding of different aspects of this proof.

What would you ask? What would s/he answer if her understanding is incomplete? How would you guide this student towards enhanced understanding? Identify several issues in this proof that may not be completely understood by a student and consider how you could address such difficulties. In your submission:

(a) Write a paragraph on what you believe could be a "problematic point" (or several points) in the understanding of the theorem/statement or its proof for a learner.
(b) Write a scripted dialogue between teacher and student that shows how the hypothetical problematic points you highlighted in part (a) could be worked out (THIS IS THE MAIN PART OF THE TASK).
(c) Add a commentary to several lines in the dialogue that you created, explaining your choices of questions and answers.

Figure 1: The Task