DIFFICULTIES IN USING VARIABLES – A TERTIARY TRANSITION STUDY

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This article describes the results obtained from a diagnostic instrument to establish the difficulties in understanding and using variables that engineering students have at the moment of their entrance to a public Mexican university that does not examine the candidates prior to admittance. This work is part of a study that analyses the possible impact that failing to understand the uses of variables may have in understanding systems of linear equations.

Key words: Uses of Variables, Simultaneous Linear Equations, 3 Uses of Variables Model, Tertiary Transition.

This study has been held in a Mexican public university that does not examine the applicants to be admitted; it only requests a High School Certificate. If the number of applicants is greater than the number of places available, the decision is taken by means of a lottery in front of a notary public. Furthermore, the students that this university receives are in most of the cases people that have interrupted their studies, from high school to university, and that come from the least favoured zones in Mexico City, where it is located.

All these preliminaries already justify the wish to know how our students understand and use variables, particularly since they are to become engineers and do not have a strong mathematical background. But our main interest focused in identifying specific difficulties in using the variables, to apply a didactic treatment based on the 3 Uses of Variables Model (3UV Model; Ursini & Trigueros, 1997, 1999, 2001) and later analyse how a rich/poor conception of variable interferes in achieving a correct mental construction of the solution to simultaneous linear equations, from the linear algebra perspective. Some researchers had suggested that not understanding the many uses of variables correctly can contribute to difficulties in understanding linear algebra (Dorier, Robert, Robinet & Rogalski, 2000), in particular the solution to a system of linear equations (Trigueros, Oktaç & Manzanero, 2007).

In this article we will focus our attention in the first part of our study, which consisted in designing a diagnostic instrument and to process the results using the 3UV Model as a conceptual framework. We show the results we obtained and the conclusions to which they led us.

Theoretical Framework and Research Methods

From the research literature we realized that we had to propose an instrument that would make it possible to identify students' common difficulties while working with variables, linear equations in one unknown and linear equations in two unknowns (all of them necessary prerequisites to the study of systems of linear equations). Some of the elements we took into consideration in our diagnostic instruments are: different factors that make literal symbols hard to understand (Wagner, 1983), arithmetic difficulties interfering with the correct solution of equations (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996) and accepting and finding multiple solutions to a linear equation in two variables (Panizza, Sadovski & Sessa, 1999). Since our main objective was to know how students performed with respect to the different uses of variables and how flexibly they could adapt to changes in the use of variables along one same problem, we chose the 3UV Model as our theoretical framework.

The 3UV Model is a theoretical framework proposed by Ursini and Trigueros (1997, 1999, 2001) as "a basis to analyse students' responses to algebraic problems, to compare students' performance at different school levels in terms of their difficulties with this concept, and to develop activities to teach the concept of variable " (Trigueros & Ursini, 2008, p.4-337). The 3UV model takes into consideration the three most frequently present uses of variable in elementary algebra: specific unknown, general number and variables in functional relationship. Its authors emphasized aspects corresponding to different levels of abstraction at which each one of the uses of variable can be handled. These aspects are described in the following paragraphs:

According to the 3UV Model, the understanding of variable as unknown requires to: recognize and identify in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem (U1); interpret the symbols that appear in equation, as representing specific values (U2); substitute to the variable the value or values that make the equation a true statement (U3); determine the unknown quantity that appears in equations or problems by performing the required algebraic and/or arithmetic operations (U4); symbolize the unknown quantities identified in a specific situation and use them to pose equations (U5).

The understanding of variable as a general number, according to the 3UV Model, implies to be able to: recognize patterns, perceive rules and methods in sequences and in families of problems (G1); interpret a symbol as representing a general, indeterminate entity that can assume any value (G2); deduce general rules and general methods in sequences and families of problems (G3); manipulate (simplify, develop) the symbolic variable (G4); symbolize general statements, rules or methods (G5).

As the 3UV Model considers it, the understanding of variables in functional relationships (related variables) implies to be able to: recognize the correspondence between related variables independently of the representation used (F1); determine the values of the dependent variable given the value of the independent one (F2); determine the values of the independent variable given the value of the dependent one (F3); recognize the joint variation of the variables involved in a relation independently of the representation used (F4); determine the interval of variation of one variable given the interval of variation of the other one (F5); symbolize a functional relationship based on the analysis of the data of a problem (F6) (Trigueros and Ursini, 2008).

As Trigueros and Jacobs (2008, p.105) recall it, "according to Trigueros and Ursini (1999, 2001, 2003) a well-developed understanding of algebra necessitates the ability to differentiate among the three uses of variable and to flexibly integrate their uses during the solution of any problem".

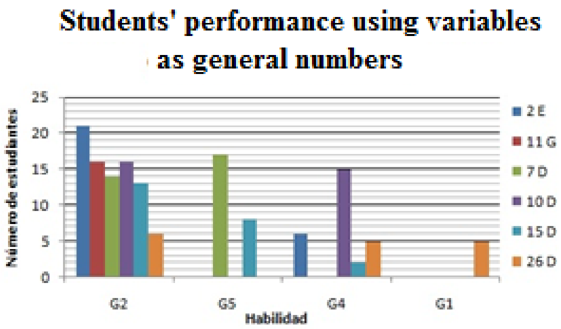
Using both, the information coming from research literature and the abilities described in the 3UV Model, the instrument questions were designed to give us insight in whether or not the students presented the difficulties described in the research literature, how students related to the different uses of variables and how flexibly they could identify a change of use in the variable in some situations. The result was a 28-question-intrument that was applied to 25 students in two 75-minute-sessions. After the first session, the students gave back the instrument together with the answers, so that at the beginning of the second session, they would continue from the point where they left the questionnaire in the previous session. The instrument was applied in the first two sessions of a first-semester-course in algebra and analytic geometry (AAG).

Preliminary results – results of the Diagnostic Instrument

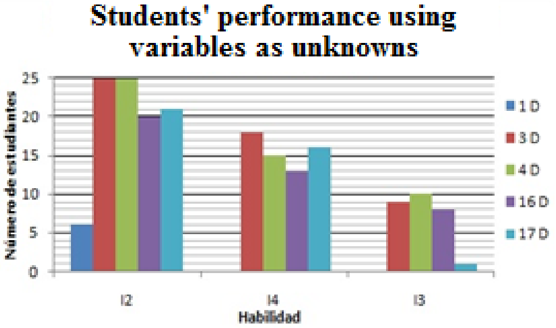
In this section we describe the results for each use of the variables and present a graph showing the general results we obtained of how students performed for each exercise of the instrument that was related to the abilities considered in the respective use of the variables being described. We decided to show general results rather than specific performance for specific exercises, to show a broader view of how rich/poor the variable conception of our students is when they enter university and to show the various aspects that would have to be taken into consideration to project a potential didactical treatment to help them enrich their conception of variable, thinking that they need to use variables fluently to solve and understand the concept of solution to simultaneous linear equations.

For all the graphs that will be presented, we show in the horizontal axis the abilities of the 3UV Model for the respective use of the variables. In the vertical axis, we show the number of students that performed correctly for the respective exercise-ability, which is represented by a bar. Each bar is labelled by the number of exercise in the diagnostic instrument, followed by a capital letter indicating which type of exercise it was. E holds for "Give an Example" tasks, G holds for "Twist" questions, D holds for Performance questions and R holds for Reflection questions, all these categories according to Zazkis and Hazzan (1999).

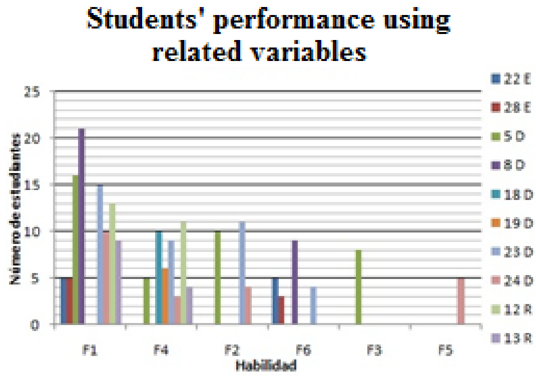
Results for the use of variables as general number: our results show that students perform better for G2, than for the rest of the abilities, but that the complexity of the exercises has a direct effect: the higher the complexity of the exercise, the lower the performance. Using exercises of different complexity results in a stronger change in performance for G4 and G5. For instance, manipulating a sequence of sums and differences that involve the variable does not represent a big challenge, but manipulating a perfect square trinomial, that requires a substitution, to rewrite it as a square binomial, already represents quite a difficult task; manipulating a sequence of operations involving variables as denominators, turned out to be an extremely difficult task. Symbolizing an open expression that involves a variable added to a number, is a relatively simple task; but symbolizing the result of a product of variables or numbers and variables is not that simple a task. In the case of G1, a regular high-school-substitution to rewrite an expression turned out to be very challenging for most of the students, who avoided it completely.



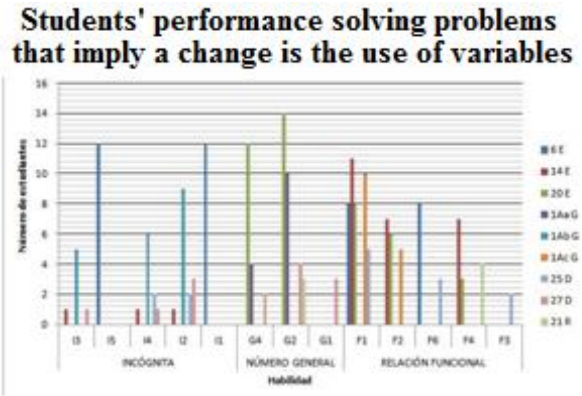
Results for the use of variables as unknowns: when presented with exercises that involve the use of variables as unknowns, students seem to be most at ease with U2, but we found that when the context is not that familiar to them, it is not clear for them when a variable is really an unknown: they tend to treat the variables in a two-variable linear equation as unknowns, not noticing that the variables are related by the equation. This is similar to what has been reported by Malisani and Spagnolo (2009) and Panizza, Sadovski and Sessa (1999). In general, we found that students consider a literal to be an unknown if it appears in an expression with an "=" sign, and that in the absence of it, they sometimes add a "= 0" to be able to manipulate the expression. Regarding U4, we found that students tend not to use algebraic procedures if it is not that difficult to solve the equation by arithmetic operations and that only in the case of equations presenting the variable on both sides of the equation they used algebraic operations from the beginning in the solution of the problem. Linchevsky and Herscovics (1996) had already detected this problem. We also found that students are not used to substitute for the variable the value or values that make the equation a true statement (U3) if they are solving the equation algebraically, as if solving it would mean to find a value through the algebraic procedure and not to find a value that satisfies it, which has also been pointed out before by Sfard and Linchevsky (1994). Another detected problem is that students tend to forget what the purpose of the problem is, and they rarely turn their attention back to the questions to check whether finding a solution was enough for solving the problem posed or if they would still need to do something else; this had been reported by Trigueros and Jacobs (2008).



Results for the use of related variables: our main result when it comes to analysing students’ performance for related variables is that with the exception of some cases in which the context is more familiar to the students, they do not know how to act when confronted with equations in related variables. For those familiar situations, they only perform relatively well for F1. They even present difficulties in determining the values of the dependent variable given the value of the independent one (F2) and vice versa (F3), if a specific value is not given for one of the variables explicitly, which coincides with what Panizza, Sadovski and Sessa had reported (1999). Not even in the case of related variables of the form “$y = k\,x$”, with k a specific explicit constant given in the problem statement, students managed to symbolize the relationship between the variables, showing how weak their F6 ability is. Recognizing the joint variation of the variables (F4) was shown not to be that difficult only in the case of very simple familiar cases (either because the problem context was a familiar situation to them or the structure was familiar to them) and when they had to reflect about someone else’s response instead of answering directly. Determining the interval of variation of one variable given the interval of variation of the other (F5) was not easy even in the case of a graphic representation of a linear equation.



Results for transitions in the uses of variables: if students have to adapt to a change in the use of a variable along the solution of the same problem, they have really strong difficulties. Of all the problems that involved a change of variable for the solution, only one student was able to solve completely most of them (and when not, it was due to arithmetic or algebraic operations). Students in general would interrupt their solution process after wandering for a while trying different things without a clear structure of what they planned to do, until they would just give up without summing-up, or they would stick to one trial and follow it until they somehow would end up with a response that they never verified to be a solution.



Concluding Remarks

We conclude from the analysed data that the students taking part in this study still have a long way to develop a rich conception of variable, and that, as Trigueros and Jacobs (2008) pointed out, “students need help in developing a rich conception”. It is necessary to help students in advancing their conceptual understanding of variables, considering the unfavourable conditions under which the students of this university have to work and not “to expect that as students encounter algebraic expressions, word problems, and problem-solving exercises, they will construct (all by themselves!) a robust, flexible and coherent conception of variable as a mathematical entity” (Trigueros and Jacobs, 2008, p.110). For the purposes of our research, now that we know in which aspects our students have a weak conception of variable, we have to first test how they perform in solving and interpreting systems of linear equations and, second, by means of analyzing the respective results with the help of the 3UV Model, make a selection of the elements that have to be strengthened to understand a system of linear equations and propose a didactic treatment accordingly. This work is in process and will be reported in future papers.

Questions to the audience

a) Considering the weak background in mathematics that our students have and that they find it very difficult to do homework due to their jobs schedules, what didactical considerations could help them in making the best use of the class time to develop the necessary abilities to enrich their concept of variable and perform better in their engineering courses?
b) What kind of short projects could be developed along a semester-course to foster the acquisition of deeper understanding of the concepts introduced in the Algebra and Analytic Geometry and Linear Algebra courses?

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