PARADOXES OF INFINITY – THE CASE OF KEN

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Previous studies have shown that the normative solutions of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation are difficult to understand even for learners with advanced mathematical background such as doctoral students in mathematics. This study examines whether this difficulty is due to the way they are set in everyday life experiences. Some variations of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation and their abstract versions set in the set theoretic language without any reference to everyday life experiences were given to a doctoral student in mathematics. Data collected suggest that the abstract versions can help learners see beyond the metaphorical language of the paradoxes. The main contribution of this study is revealing the possible negative effect of the metaphorical language of the paradoxes of infinity on the understanding of the learner.

Keywords – Infinity, Paradoxes, Cognitive conflict

Introduction and Theoretical Perspectives

Paradoxes involving infinity have been used as a lens in mathematics education research for identifying students' difficulties in understanding infinity. One study was conducted by Mamolo and Zazkis (2008) who used the paradoxes Hilbert's Grand Hotel and the Pin-Pong Ball Conundrum. This study is a part of PhD thesis research of Mamolo (2009) which also included the Pin-Pong Ball Variation. In Mamolo (2009) the author reports that even students with advanced mathematical background including some doctoral students in mathematics had trouble understanding the normative solutions of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation. This study examines whether this difficulty in understanding the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation is due to the way these paradoxes are set in everyday life experiences.

What kind of thinking involves in understanding paradoxes like the Pin-Pong Ball Conundrum? Barbara, Dubinsky and McDonald (2005) suggest that it is advanced mathematical thinking. They define Advanced Mathematical Thinking as thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses. They say that comparing $|N|$ with $|2N|$ may require Advanced Mathematical Thinking and the ability to understand that there is a one-to-one relationship between N and 2N is probably not available through experience in the physical world.

We consider two other theoretical frameworks to analyze the data. One is APOS analysis of conceptions of infinity by Dubinsky, Weller, McDonald, and Brown (2005). They suggested that interiorizing infinity to a process corresponds to an understanding of potential infinity - infinity is imagined as performing an endless action. The ability to conceive of the process as a totality occurs as a consequence of encapsulation of the process to an object, and corresponds to a conception of actual infinity.

The other theoretical framework is reducing abstraction by Hazzan (1999). According to this perspective abstractness of mathematical concepts can be reduced by connecting them to real-life situations and establishing a right relationship (in the sense of Wilensky) between the learner and the mathematical concept.

Research Method

Ken is a PhD student in mathematics at a big ten university in the Midwest in the USA. He was sent the following questions in a questionnaire and later interviewed after getting his answers to the questionnaire.

1. A large barrel has Pin-Pong balls numbered 1, 2, 3 … The following task is done in one minute. In the first half of the minute the ball number 1 is removed. In half the remaining time the ball number 2 is removed. Again, in half the remaining time the ball number 3 is removed, and so on. At the end of the minute, how many Pin-Pong balls remain in the barrel?

2. A large barrel has Pin-Pong balls numbered 1, 2, 3 … The following task is done in one minute. In the first half of the minute the ball number 1 is removed. In half the remaining time the ball number 11 is removed. Again, in half the remaining time the ball number 21 is removed, and so on. At the end of the minute, how many Pin-Pong balls remain in the barrel?

3. Let $A_n = A_{n-1} - \{n\}$ for n = 1, 2, 3, … where A_0 is the set of positive integers. Describe $\bigcap_{n=1}^{\infty} A_n$.

4. Let $A_n = A_{n-1} - \{10(n-1)+1\}$ for n = 1, 2, 3, … where A_0 is the set of positive integers. Describe $\bigcap_{n=1}^{\infty} A_n$.

Questions 1 and 2 are variations of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation, given in the appendix with their normative solutions, respectively with the same end result but different processes. For example in Question 1 the number of balls in the barrel decreases and is always infinite as time approaches the end of one minute but in the Pin-Pong Ball Conundrum the number of balls increases and is finite as time approaches the end of one minute. Questions 3 and 4 are the abstract versions of 1 & 2. Abstract versions don't have a sense of time.

Results

He answered all the questions correctly. From what he wrote at the end of the questionnaire it is clear that he saw that Questions 3 & 4 formalize the processes described in Questions 1 & 2 respectively. And Question 2 helped him in Question 4. So he clearly saw the connection between the concrete versions and the abstract versions. He also reduced the abstraction in Question 4 by going back to Question 2. Even though, arguably Ken is capable of Advanced Mathematical Thinking, he had trouble understanding the processes in Questions 1 and 2. The abstract versions helped him to see that the process can be continued.

Researcher: what if you did not get number 3 and 4 and you got only 1 and 2?

Ken: yeah then … I would still probably I need to take more time I will probably end up assuming that I have to think that this process can be done and I would still give the same answer but after I mean it take bit more time to kind of assume that to take that.

So without Questions 3 and 4 he thinks he would have answered Questions 1 and 2 the same way but it would have taken him more time. Ken never questioned the plausibility of the Questions 3 and 4. As an advanced graduate student in pure mathematics he knows the mathematical language well. He can work in the mathematical realm. So he did not have any trouble with Questions 3 and 4. Though he interiorized the action of removing the ball number n in Question 1 as a process he could not encapsulate this process to an object:

Ken: I started from the first question but I didn't write down answers because at some point I was little bit confused about problem 1 because since it was kind of a practical procedure although it was clear what was going on I mean specially answering the last part

APOS analysis can be applied to Questions 3 and 4 as well. Apparently Ken did not have any trouble with encapsulating the intersection of infinitely many sets to an object – he got little help from Question 1 and 2 in describing this object.

Discussion & Conclusions

Paradoxes involving infinity can provide a window to infinity. The cognitive conflict elicited by a paradox is difficult for a learner to resolve. Resolving this cognitive conflict requires the learner to make a cognitive leap from the intuitive to the formal or from the real world to the mathematical realm. But some of the paradoxes make this cognitive leap difficult as they are too far away from the reality but yet set in the everyday life experiences. If we compare Zeno's paradox of Achilles and Tortoise and the Pin-Pong Ball Conundrum, we can see that Achilles and Tortoise is about a real life situation and the Pin-Pong Ball Conundrum is not a real life situation though it involves real life objects. Even in the mathematical realm the concept of infinity is a difficult concept to grasp. Bolzano and Galileo could not grasp infinity though they considered abstract mathematical entities like intervals and sets of numbers. So when the concept of infinity is presented through everyday life experiences with an infinite process in a finite time interval far away from reality it adds to the difficulty of grasping infinity. We can see it from Ken. Our findings agree with Mamolo (2009) who found that even students with advanced mathematical background including some doctoral students in mathematics had trouble understanding the normative solutions of the Pin-Pong Ball Conundrum and the Pin-Pong Ball Variation. There is further evidence in Mamolo and Zazkis (2008): "Based on the results of our research, and specifically acknowledging the similarity in responses of students with different mathematical sophistication, we suggest that a formal mathematical view of infinity implied in conventional resolutions of the paradoxes may not be reconcilable with intuition and 'real life' experience."

The concept of infinity in mathematics is very mathematical and counter intuitive. This study reveals that the metaphorical language of the paradoxes could have a negative effect on the understanding of the learner.

Questions for the Audience

1. Tasks in Questions 1 & 2 are the same in some sense: Always in half the remaining time a ball is removed. But at the end of the minute the outcomes in them are very different. In the first task the barrel is empty and in the other it has infinitely many balls. Can this happen in the physical world?
2. Can we imagine an infinite process where each step takes some time in a finite time?
3. Is it effective to teach mathematical concepts that are counter intuitive and do not relate much to the experience in the physical world using a real life context?

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Appendix

The Pin-Pong Ball Conundrum

An infinite set of numbered Pin-Pong balls and a very large barrel are instruments in the following experiment, which lasts one minute. In the first half of the minute, the task is to place the first 10 balls into the barrel and remove the ball number 1. In half the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the minute, how many Pin-Pong balls remain in the barrel?

Solution

In this thought experiment there is an infinite sequence of time intervals of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$. Since in the time interval of length $\frac{1}{2^n}$ the ball number *n* is removed and there is a one to one correspondence between the sets $\{\frac{1}{2^n} / n \in N\}$ and *N*, the set of positive integers, at the end of the minute the barrel is empty.

The Pin-Pong Ball Variation

An infinite set of numbered Pin-Pong balls and a very large barrel are instruments in the following experiment, which lasts one minute. In the first half of the minute, the task is to place the first 10 balls into the barrel and remove the ball number 1. In half the remaining time, the next 10 balls are placed in the barrel and ball number 11 is removed. Again, in half the remaining time, balls numbered 21 to 30 are placed in the barrel, and ball number 21 is removed, and so on. After the experiment is over, at the end of the minute, how many Pin-Pong balls remain in the barrel?

Solution

In this variation the same infinite sequence of time intervals of length $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ is there. But in the time interval of length $\frac{1}{2^n}$ the ball number $10(n-1)+1$ is removed. And there is a one to one correspondence between the sets $\{\frac{1}{2^n} / n \in N\}$ and $\{10(n-1)+1 / n \in N\}$. So at the end of one minute the barrel has the balls numbered 2, 3, … , 9, 10, 12, 13, … , 20, 22, … - this corresponds to the set $N - \{10(n-1)+1 / n \in N\}$.