INTERPLAY BETWEEN CONCEPT IMAGE & CONCEPT DEFINITION: DEFINITION OF CONTINUITY

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This study looks at the interplay between the concept image and concept definition when students are given a task that requires direct application of the definition of continuity of a function at a point. Data was collected from 37 first year university students. It was found that different students apply the definition to different levels, which varied from formal deductions (based on the application of the definition) to intuitive responses (based on rather loose and incomplete notions in their concept image).

Keywords – Continuity, Concept image, Concept definition, Concept definition image, Cognitive conflict

Among others, functions, limit, derivative and continuity have been widely recognized as some of the advanced mathematical concepts that not only students but also teachers find somewhat hard to grapple with. In addition to research carried out on the understanding of these concepts individually (Bezuidenhout, 2001; Vinner, 1987; Cornu, 1991), there has also been research done on understanding of the relationships between some of these concepts (Aspinwall et al., 1997; Duru et al., 2010). Further, the presentation of these concepts in a particular text book is discussed by Tall & Vinner (1981). This paper aims to look at how students work with the concept of continuity. Concept image and concept definition by Vinner (1991) will serve as a theoretical framework for the analysis of the data. This study is driven by the following questions: To what extent do students recall and apply the definition of continuity when handling tasks involving continuity? What notions of continuity are present in their concept images?

Research Method

Thirty seven student responses to the following question were collected and analyzed for this study.

Let

$$f(x) = \begin{cases} \frac{x^2 + x - 2}{x - 1} & ; x \neq 1 \\ a & ; x = 1 \end{cases}$$

Which value must you assign to a so that $f(x)$ is continuous at $x = 1$?

The students were in their first year of undergraduate studies specializing in the biological and medical sciences and were taking a Calculus course. They had covered the topics functions, limits, limit laws and continuity at the time of data collection. Based on the definition that was taught in this course "A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$", a complete answer to the above question may include three distinct points.

- Identifying the condition that must be satisfied for $f(x)$ to be continuous at $x = 1$. For $f(x)$ to be continuous at $x = 1$, $\lim_{x \to 1} f(x)$ must be equal to $f(1)$ which is a.
- Finding the limit of $f(x)$ when x approaches 1.

$$\lim_{x \to 1} f(x) = \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)} = \lim_{x \to 1} x + 2 = 1+2 = 3 \quad ; x \neq 1$$

- Concluding that a must be 3.

$f(1) = 3$ and $f(1) = a$ hence $a = 3$.

These steps need not be in this same exact order but there must be some logical sequence in the way the students organize their answer. The consultation of the definition in the first step requires them to proceed to the second step where they need to find the limit of $f(x)$ when x approaches 1. A student may do this step first 'knowing' it needs to be done in their head and may state the condition afterwards. Because without calling on the definition, there will not be a necessity to find the limit. The second step is a matter of finding the limit of a function where the function is a rational which produces an indeterminate form with direct substitution. This step hence, may not call on the definition of the limit but only on the procedures of finding the limit. Last step is the conclusion of the answer.

Results

Four different types of answers could be identified. The four categories are listed in a certain order which is from a poor answer to a good answer from a marker's perspective.

Type 1 - *The correct answer for a is obtained but taking the limit of $f(x)$ when x approaches 1 is not explicitly shown.*

Four (out of 37) students in the group gave the answer in this category as shown in figure 1. It is hard to say whether these students are *thinking* of taking the limit but not showing it or they are merely doing an algebraic manipulation of the expression. The line, $f(1) = ((1) + 2)$ can be interpreted at least in two ways.

Case 1 : 'plugging a value into the function'

$f(x) = (x + 2)$ and hence $f(1) = ((1) + 2)$ or

Case 2 : applying the condition for continuity and hence stating an identity

$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 2) = ((1) + 2)$ & this must equal to $f(1)$, $f(1) = ((1) + 2)$

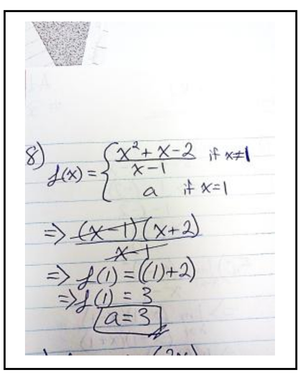

Figure 1: Type 1

The way they have presented their answer it appears as though the students meant the first case rather than the latter. This is because if they meant the second case, the way the argument is ordered, it should be written as $((1) + 2) = f(1)$, not as $f(1) = ((1) + 2)$.

The concept definition of 'continuity of a function at a point' *contains* the concept of 'limit of a function'. If students have trouble understanding the concept of limit and hence possess a blurred concept image of limit, then, this has a significant impact on the concept image of continuity. The portion of their concept image which is evoked by this problem does not seem to contain or have any overlap with the concept of limit. Their working can be best described as an effort to merge the two pieces of the function. This can be pointing to the notion that students were found to have by Tall and Vinner (1981) too, of the need for a function to be in *one piece* to be continuous. It appears that they simplify the case when $x \neq 1$ which is $\frac{x^2 + x - 2}{x - 1}$ to $(x + 2)$ and then assign it to the case when $x = 1$.

The features of the responses of this type also suggest that the task has not made these students to consult the concept definition but that they have worked on certain notions in their concept image of continuity. This intuitive response is modeled by figure 2 as illustrated by Vinner (1991, pg. 73).

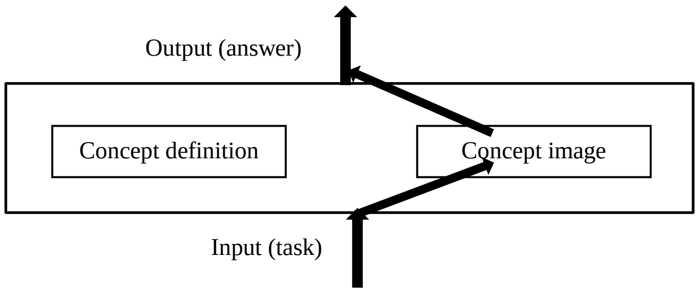

Figure 2: Intuitive response

Type 2 – *The limit is taken and the value for a is given without noting that the limit must equal to $f(1)$.*

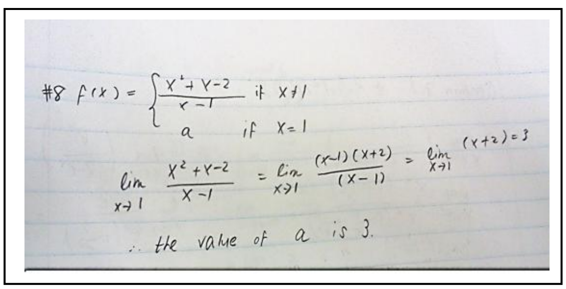

Figure 3: Type 2

In this category (12 out of 37) the students have taken the limit and have just concluded that it is equal to a (see figure 3). This kind of an answer can come from a correct reference to the definition. What is lacking in terms of writing is, not explicitly showing or stating that the calculated limit must equal the function value at $x = 1$. And it is not acknowledged that $f(1) = a$. However, this may have been thought through to obtain the answer as $a = 3$.

Another possible process that may be on work here is a rote memorization of a procedure rather than any attention given to the definition. Since this is a familiar and 'routine' kind of question, students may have developed an algorithm for it, as part of the concept image. It may be a rule like 'find the limit of the function given and assign it to the letter'. Only this procedure, in that case, may be evoked when presented with this style of a question.

Type 3 - *The limit is taken and notes that it should be equal to $f(1)$ and hence to a.*

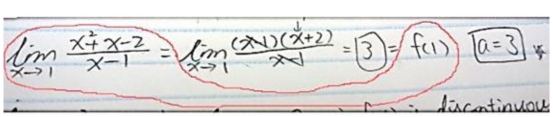

Figure 4: Type 3

These students (4 out of 37) have explicitly stated that $f(1)$ is equal to the answer they obtained for the limit and hence have exhibited an important part of the definition before concluding the final answer for a (see figure 4). And as shown in figure 4, the definition is embedded in their answer. It can be concluded that, in their concept image they have a complete concept definition image which they have been able to appropriately apply in this task. Based on the presented written work, students in this type are a step ahead of the students under type 2. Even if one argues that these students too can be applying a mere memorized algorithm, it is evident that their 'algorithm' is more closely grounded to the definition.

Type 4 – *A complete logical answer with all reasons is given.*

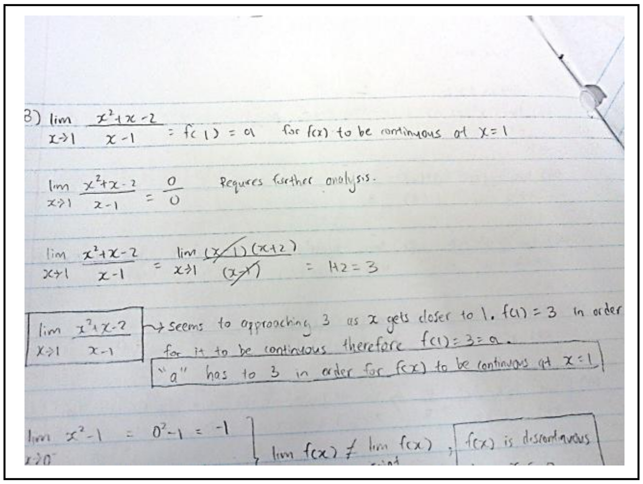

Figure 5: Type 4

The answers were with a good logical sequence of reasoning without missing any points as shown in figure 5. Thirteen of the students had given answers in this category.

It is clearly demonstrated how they formulate their answers by consulting the concept definition. And no sign of side tracking or being disturbed or intervened by unnecessary notions that *may* be present in the concept image is visible. Hence, this can be modeled by figure 6 as illustrated by Vinner (1991, p. 72) of a purely formal deduction.

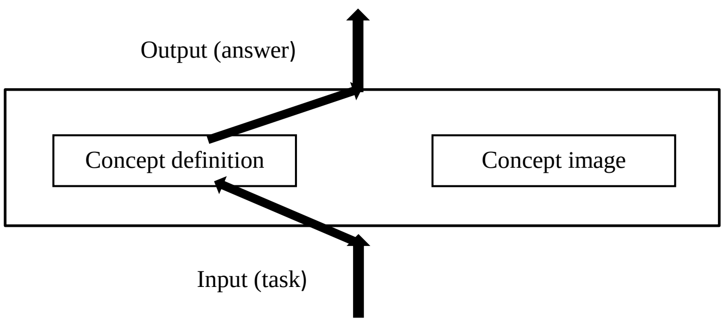

Figure 6: Formal deduction

Discussion & Conclusions

Response types 2,3 and 4, show clear attention given to the definition in different degrees. Vinner(1991) claims that the majority of students do not use definitions when working on cognitive tasks in technical contexts and that college courses do not develop in the science students, not majoring in mathematics, the thought habits needed for technical contexts. However, as far as using definitions goes, this study suggests that, majority of students who are not majoring in Mathematics do refer the definition but in different levels. They seem to have a concept definition image developed to different levels as part of their concept images. Or, if the assumption- that their writings reflect their cognitive processes- is removed, this can be pointing to a different category of levels in transforming their cognitive processes into writing.

What seems to emerge from type 1 is the tendency of some students to tackle problems in ways that they have built for themselves with little rigor which works and produces the correct answer. Vinner says that ‘as long as referring to the concept image will result in a correct solution, the student will keep referring to the concept image since this strategy is simple and natural’ (Vinner, 1991, pg. 80). Can this be overlooked as they produce the correct answer and be satisfied about their performance, as these students are not majoring in Mathematics? Or should these be resolved by creating cognitive conflicts that make students confront these erroneous methods?

Discussion Questions

1. Are there any other ways in which you can interpret student thinking/reasoning corresponding to the type 1 (figure 1) answer?
2. The text book uses the technique of cancellation of factors in examples and does not mention that what is obtained after cancelling the common factor is a different function that agrees with all but one point of the original function. What effect does - not knowing what is going on behind this technique - have on future learning of students, if any? How important is it for students to know this?
3. What other kinds of questions would be more effective in finding out erroneous concept images of continuity in students?

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