IN-SERVICE SECONDARY TEACHERS' CONCEPTUALIZATION OF COMPLEX NUMBERS

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This study explores in-service high school mathematics teachers' conception of various forms of a complex number and the ways that they transition between different representations' (algebraic and geometric) of these forms. Data were collected from three high school mathematics teachers via a ninety- minute interview after they completed professional development on complex numbers. Results indicate that these teachers do not necessarily objectify exponential form of complex numbers and only conceptualized it at the operational level. On the other hand, two teachers were very comfortable with Cartesian form and showed process/object duality by translating between different representations of this form. It appeared that our participants' ability to develop a dual conception of complex numbers was bound by their conceptualization of the various forms, which in turn was hindered by their representations of each form.

Key words: Complex numbers, In-service secondary teachers, Operational/structural conceptualization, Representations

Introduction

Understanding the complex number system, including performing arithmetic operations with complex numbers and representing complex numbers and their operations on the complex plane, is one of the Mathematics standards for high school highlighted in the Common Core State Standards Initiative (CCSSI, 2010 Appendix A, p. 60). The document emphasizes the need for students to work with multiple representations of complex numbers (e.g., algebraic and geometric), and also recommends that students know how to represent complex numbers using rectangular and polar forms. In order for students to develop these notions, it is necessary for teachers to have deep content knowledge as well as knowledge about teaching the field of complex numbers. Deep content knowledge of the field of complex numbers entails knowing the multiple representations and forms, understanding the connections among them, translating between forms flexibly, and recognizing which representations and forms would be more suitable to use in a given task. However, teachers' understanding of complex numbers as well as the required pedagogical content knowledge has been understudied in mathematics education. Investigating this phenomenon is natural given that representations have played a significant role in the field of mathematics education (e.g., Eisner 2004; Janvier 1987; NCTM 2000) and the fact that multiple representations are an integral characteristic of complex numbers.

The purpose of this report is to share our findings from a research study conducted with secondary mathematics teachers that investigates teachers' content knowledge of complex numbers. In this paper we address the following research question: How do secondary mathematics teachers conceptualize complex numbers and their arithmetic operations? More specifically we explore teachers' conception of different representations and forms of complex numbers.

Literature Review

Mathematics educators typically insist that learners should have access to multiple representations in order to reveal information or to illustrate solutions to problems or mathematical ideas. One reason for the emphasis on multiple representations is that "Different representations often illuminate different aspects of a complex concept or relationship" (NCTM, 2000, p.69). Research studies stress the importance of using multiple representations and indicate that learners develop mathematical reasoning and better conceptual understanding when they use various forms of representations (Cuoco & Curcio, 2001). In other words, learners improve their ability to think mathematically by engaging with multiple representations.

Although there is a vast amount of literature related to the role that representations play in the arithmetic of real numbers (Kilpatrick, Swafford, & Findell, 2001; Sowder, 1992), the same is not true for complex numbers. A handful of researchers (Conner et al., 2007; Danenhower, 2000; Nemirovsky et al., 2012; Panaoura, et al., 2006) have begun to explore students' as well as experts' (Soto-Johnson et al., 2011; Soto-Johnson et al., 2012) geometric interpretations of complex numbers and complex valued functions. In general, findings indicate that the experts effortlessly regarded complex numbers and their operations and complex valued functions as dynamic objects. However, similar results were not observed with students.

Conner et al. (2007), found that prospective secondary mathematics teachers perceived multiplication of a real number by -1 as a reflection rather than a rotation of $180°$. This perception may be a result of focusing on the real number line rather than the entire complex plane and may have contributed to their inability to illustrate how multiplication by the complex number $x + yi$ results in a rotation and a dilation of the other factor. The preservice teachers also described complex numbers as a pair of real numbers rather than as a single number. This perspective may have facilitated their ability to provide a geometric interpretation of the addition of complex numbers using vectors or by decomposing the numbers into the real and imaginary components. Unfortunately, this perception about complex numbers does not lend itself to the geometric interpretation of complex number multiplication. Nemirovsky et al. (2012), however, offered methods for fostering a dynamic view of multiplication by i. As part of a teaching experiment with preservice secondary teachers, where the classroom floor served as the complex plane, the participants physically engaged in exploring the behavior of multiplying $2 + \frac{1}{2}i$ by i. Such perceptuo-motor activity provided a setting where the participants discovered and conceptualized the structural components behind complex number addition and multiplication.

In another related study, Danenhower (2006) asked undergraduates to convert instantiations of the fraction $\frac{a+bi}{c+di}$ into Cartesian $(x + yi)$ or exponential $(re^{i\theta})$ form. The participants tended to avoid the exponential form, especially if it required converting between polar $(r(\cos\theta + i\sin\theta))$ and exponential forms due to their perceived weakness with trigonometric functions. Although the students easily worked with the Cartesian form, it was generally not the most efficient method to simplify the fraction. In many instances a geometric interpretation would have alleviated much of the computational effort, but the participants did not appear to draw on such an interpretation. Similar results were found by Panaoura et al. (2006), who investigated Greek high school students' ($N = 95$) proficiency moving between algebraic and geometric representations of complex-valued equations and inequalities of the form $|z - z_0| \le k$. In general, the participants were more successful when explicitly asked to provide an algebraic representation of a given geometric figure, but this was not evident when the students were asked to solve a similar problem-solving task. This may suggest "a lack of flexibility in using the geometric approach effectively with different representations of complex numbers" (p. 700).

While the research suggests that high school students and undergraduates tend to view i as a static object, struggle to provide geometric meaning to complex number arithmetic, and fail to recognize which form is more appropriate for a given situation, Soto-Johnson et al. (2011) and Soto-Johnson et al. (2012) indicates that experts do not have such difficulty. In attending to given tasks, the experts easily recognized which representation was better suited for each task. It was "natural" for them to view the complex numbers as vectors for addition, to represent them in Cartesian form for addition and polar form for multiplication, and to express z as an operator and w as the operand for multiplication. The experts easily connected algebraic and geometric representations and navigated between representations and forms. Furthermore, they recognized which form and which representation were most appropriate for more advanced tasks. This flexibility allowed them to provide responses involving metaphors, which highlight the dynamic aspects of complex numbers.

Theoretical Perspective

In an effort to explore teachers' conception of complex numbers, we incorporated Sfard's (1991) duality principle of conception for the different representations and forms of complex numbers. Sfard defines *conception* as "the whole cluster of internal representations and associations evoked by the concept [or notion]" (1991 p.3), and describes two types of conception: *operational* and *structural*. *Structural conception* refers to treating or seeing mathematical notions as abstract objects. An example includes perceiving a complex number as a number- a fully-fledged mathematical object on which processes can be performed. *Operational conception* focuses on the "processes, algorithms and actions" (p.4) performed on mathematical notions. For example, recognizing i as the square root of negative one. When there is no evidence of conception, Sfard (1991) classifies this as the pre-conceptual stage. The two conceptions, *operational* and *structural*, of the same notion complement each other and foster a dual conception.

In the development of a mathematical notion, operational conception precedes structural and three stages of development, *interiorization*, *condensation* and *reification*, illustrate the transition from process to object. During the stage of *interiorization* the learner skillfully performs processes on developed mathematical notions. At the stage of *condensation* the learner can perform many processes and is capable of viewing them as a whole without going into details of each step. As learners progress in this stage, they begin to manifest more flexibility in translating between different representations of the same notion. The progression continues until the learner starts to recognize the object as a new entity or is able to distinguish the object from the processes. *Reification* is the stage when the learner can extract the object from processes. In contrast to the previous stages, the shift to reification can be instantaneous. At this stage different representations of the same notion merge together.

In our study, our participants were introduced to the Cartesian, polar, and exponential forms of a complex number with algebraic and geometric representations for each form. We incorporated Sfard's (1991) duality principle as part of the professional development highlighting the connection between different representations to provide an opportunity for the teachers to condense and possibly reify the different forms of complex numbers. Sfard highlights the importance of such practices and warns "As long as the computational processes have been presented in the purely operational way, they could not be squeezed into static abstract entities, thus were not susceptible of being treated as objects." (p.24) We examined the three teachers' dual conception of each form to capture their overall conception of complex numbers. In our analysis, we explored participants' use of multiple representations with various forms in order to distinguish between their operational and structural conception of a form.

Methods

As part of this study, in-service high school mathematics teachers engaged in a three-day professional development (PD) program intended to strengthen their content knowledge of complex numbers. Besides introducing the participants to the three forms of a complex number, we also illustrated various representations for each form. As part of the PD the teachers engaged in discussions emphasizing the connection between these various forms, shared their perceptions of complex number arithmetic, discovered dynamic representations of the arithmetic of complex numbers using *GeoGebra*, compared real and complex number arithmetic, and provided algebraic and geometric explanations for "complex sentences."

Three teachers, Melissa, Aaron, and Troy (all pseudonyms), participated in an individual 90-minute task-based interview after the completion of the PD. The goal of the interview was to gain understanding of the ways in which the teachers used different representations in their mathematical reasoning of complex numbers with novel tasks presented in various forms. At the time of our study Melissa was in her first year as a full-time teacher and taught algebra II and geometry; Aaron taught geometry and was in his second year of teaching; and Troy, who was in his 21st year of teaching, taught IB mathematics. Both Aaron and Troy had Masters degrees in mathematics education. These interviews served as our primary source of data; other data included video-recording of each of the PD days and teachers' in-class work, which was used for triangulation purposes. All three interviews were fully transcribed and each member of the research team used deductive analysis techniques (Erickson, 2006) to code the teacher's responses. This entailed cataloging how and when the participants used various representations for each form. This allowed us to provide evidence regarding the dual conceptualization of a given form. We refined our results after sharing and discussing our individual analysis, which was followed with a cross-case analysis.

Results

Our analysis suggests that none of the interview participants (Melissa, Aaron, and Troy) had a dual conceptualization of a complex number, although each teacher articulated reasoning that conveyed structural conceptualization for some forms of complex numbers. In other words, our participants' conceptualization of a complex number tended to be bound by their conceptualization of each form.

Overall, Melissa had an operational conception of i, while evidence suggests both Aaron and Troy had a dual conception of i. During the interview Melissa referred to i as the square root of -1 multiple times and utilized the fact that i^2 is equal to -1 in her explanations of her solutions. Even though she recognized and used different representations of i, she did not flexibly connect the various representations. For example, when she wanted to represent i as a point on the Argand plane, she was hesitant whether it was the point $(0,1)$ and asked the interviewer if her point was correct. After receiving confirmation she was not hesitant anymore. Throughout the remainder of the interview she translated back and forth between a point representation and algebraic one as she performed manipulations with i. We interpreted Melissa's such actions as her trying to condense the form i by moving between representations. However, we did not find any evidence where she reified this particular form. For these reasons we believe that Melissa had an operational conception of i and appeared to be at the condensation stage. On the other hand, while Aaron and Troy both stated and used the fact that i was the square root of -1, they both also recognized and utilized different representations of i flexibly in their explanations and solutions during the interview.

Similarly, for the Cartesian form Melissa had an operational conception, while Aaron and Troy had a dual conception. When asked to describe how she thinks of a complex number, Melissa replied with "*Well I guess just the letter* i *and anything that correlates with having* i*, so like* $i + 1$ *and multiples of* i *and all that…"* She used this description of a complex number throughout the interview. Melissa relayed a complex number in Cartesian form as an algebraic process performed on i, which is evidence of an operational conception of this form. In contrast, the evidence suggests that a complex number in the form $a + bi$ is an object for both Troy and Aaron. At one point in the interview, Aaron stated "*So if you're telling me* z *is complex figure,* z *is going to be in the form* $a + bi$*,"* and a similar instance occurred with Troy. Such instances were coded as a structural conception of this form, since the participants used this form as an object. The reason for such coding decisions were from

Sfard's framework in which she suggests that "when tackling a genuinely complex problem, we do not always get far if we start with concrete operations; more often than not it would be better to turn first to the structural version of our concepts." (p.27) Moreover, both Aaron and Troy were able to consider multiple representations of the Cartesian form simultaneously. For example, while working on an interview task, they each declared that multiplying a complex number by i took the point (a, b) to the point $(-b, a)$.

During the interview, Melissa was not able to work effectively with the exponential form. For example, when asked to explain why $\frac{r_1 e^{i\phi}}{r_2 e^{i\theta}} = \frac{r_1}{r_2} e^{i(\phi-\theta)}$ was true, Melissa responded that her solution method was "*comparing both sides of the equals [sign]*". She continued to articulate how she would simply compare the symbols and use the "law of exponents", which she had just covered in class. Such a response led us to believe that her conceptualization of the exponential form was at the pre-conceptual level. On the other hand, both Troy and Aaron appeared to possess an operational conceptualization with the algebraic representation of the exponential form. This was evidenced with their quick response that the statement was true due to the law of exponents. Furthermore, Troy demonstrated a pre-conceptual level of the geometric representation for exponential form (a polar vector representation of exponential form was considered to be a geometric representation of this form), while Aaron provided evidence of a pseudostructural conceptualization for the geometric representation of the exponential form. These conclusions are based on the fact that Troy struggled to recall the meaning of the exponential form and his attempt to divide vectors was problematic, which seemed to suggest he possessed a preconceptual understanding of the exponential form. Similarly, Aaron viewed the complex numbers as vectors as evidenced in his statement, "*Or you can think of it as vector 1 being divided by vector 2. So vector 2 is acting on vector 1."* The fact that Aaron perceived the task as division of vectors, which does not make mathematical sense, appeared to hinder his ability to provide a viable geometric representation illustrating his algebraic explanation. Sfard (1992) provides a special term for such a case, pseudostructural conception, meaning that a person has both operational and structural conceptualization at certain instances and neither at other cases, and states "such tendency may indicate a semantically debased conception." (p.75) This led us to believe that Aaron had a pseudostructural conception of the geometric representation of the exponential form of complex numbers.

Conclusion

Our results indicate that the participants did not have dual conception of complex numbers, however developed duality of some forms of complex numbers. Even though teachers were provided opportunities during PD to condense and reify complex numbers structurally by practicing using various forms and translating between them using various representations, such practices were only observed in certain forms of complex numbers. As Sfard (1991) states "The reification, which brings relational understanding, is difficult to achieve, it requires much effort […]." (p.33). It is quite possible our participants needed more time to reify the complex numbers.

Our experiences make us believe that universities need to examine how they train prospective teachers and offer PD for inservice teachers regarding complex numbers. We are not proposing that preservice teachers complete a complex variables course, but room must be made in the curriculum for prospective teachers to develop a dual conceptualization of complex numbers. For example, such exposure could exist in methods and technology courses designed specifically for prospective secondary mathematics teachers. In the methods course, opportunities could be provided for the prospective teachers to review high school texts in order to obtain a better idea of where and how complex numbers emerge in the curriculum. A technology course is another excellent venue where both preservice and inservice teachers can learn about complex numbers using software such as *Geometer's Sketchpad* or *GeoGebra* to explore the behavior of complex-valued functions. Such practices may reinforce the progress towards a dual conceptualization of complex numbers.

The recommendations put forth by the Common Core State Standards for high school students to understand the structure and properties of complex numbers as well as their arithmetic operations will hopefully transform how complex numbers are taught in high school level. But such a transformation will also require assistance from schools, universities, and assessment agencies. More research investigating both students and teachers' conceptualization of complex numbers and complex valued functions will help us to develop better teaching practices in this content domain.

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