SYSTEMATIC INTUITIVE ERRORS ON A PROVE-OR-DISPROVE MONOTONICITY TASK

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Despite the importance of intuition and analysis in proof tasks, students have various difficulties with both types of reasoning. Such difficulties may be attributed to insufficient intuition, logical reasoning skills, or concept images. However, dual-process theory asserts that intuition can form inaccurate or incomplete representations of tasks based on systematic errors before analysis can respond. Thus, students' difficulties may be attributed to systematic intuitive errors rather than inadequate intuitive or analytical reasoning. In this study, I conducted task-based interviews with four undergraduate and one graduate mathematics major in which they completed prove-or-disprove tasks. In this paper, I discuss the systematic intuitive errors committed by these students on a monotonicity task. These errors led all five students to believe incorrectly that the statement in the task was true. Furthermore, each student engaged in correct mathematical reasoning guided by their incorrect intuitive representations.

Key words: Reasoning and proof, Intuition, Dual-process theory, Task-based interviews

Intuition and analysis are fundamental components of mathematics that play key roles in producing proofs and counterexamples (Fischbein, 1987; Tall, 1991; Wilder, 1967). However, undergraduate students have a multitude of difficulties with both types of thinking that inhibit effective reasoning on proof tasks. Intuitive difficulties include (a) lack of intuition (Moore, 1994) and (b) narrow intuitions based on examples and visualizations (Moore, 1994; Raman, 2003). Analytical difficulties include (a) limited logical reasoning skills (Harel & Sowder, 2007; Selden & Selden, 1987) and (b) incomplete or inaccurate concept images (Tall & Vinner, 1981). Furthermore, students have difficulties connecting their intuitive understandings to analytical arguments. Students' intuition may not lead directly to a proof or counterexample, or students may not recognize the relationship between their intuition and a proof or counterexample (Raman, 2003). This may result in students' inability to understand formal mathematical statements or begin a proof (Moore, 1994).

Dual-process theories of reasoning assert that intuition and analysis correspond to distinct types of cognitive processing, each with specified characteristics and roles (Evans, 2006, 2008, 2010; Kahneman, 2002). Although dual-process theories result from cognitive psychology research, Leron and Hazzan (2006, 2009) have suggested using them to analyze recurring errors in mathematical tasks. Dual-process theories suggest that certain systematic errors that recur across tasks and participants can be attributed to flawed intuitive reasoning that steals the show before analytical reasoning even takes the stage.

Preliminary results from a study in which students decided whether to prove or disprove a mathematical assertion and constructed corresponding proofs or counterexamples indicate that students' errors may be attributed to systematic intuitive errors. Furthermore, students' analytical reasoning was incorrect only because it was based on these errors.

Theoretical Framework

Dual-process theory asserts that reasoning uses two distinct types of cognitive processes – intuitive and analytical (Evans, 2008). Although these processes collaborate, intuition often dominates and influences analysis in unproductive ways (Evans, 2010).

Intuition is often quick, automatic, requires little cognitive effort, and is developed through experience (Evans, 2008; Fischbein, 1987; Wilder, 1967). The automatic operation of intuition frequently provides a default response to a task (Evans, 2010; Wilder, 1967) that takes into account prior knowledge and beliefs, task features, and the current goal of the reasoning to create a representation of the task (Evans, 2006). However, the representation may be distorted or deficient due to systematic accessibility errors of intuitive reasoning (Evans, 2010; Kahneman, 2002). *Accessibility* is the ease with which certain knowledge is evoked or certain task features are perceived (Kahneman, 2002). Two key accessibility errors involve (a) attribute substitution, and (b) knowledge and task feature relevance.

Attribute substitution errors occur when a more easily accessible attribute is substituted in a task for a less easily accessible attribute (Kahneman, 2002). Participants often intuitively notice similarities between the current task and previously encountered tasks, substitute accessible attributes for less accessible ones based on these similarities, and unknowingly change the given task to a similar more accessible task (Kahneman & Frederick, 2002).

Relevance errors occur when knowledge and task features are deemed irrelevant because they are not readily accessible (Evans, 2008, 2010; Kahneman, 2002). When forming intuitive task representations, (a) less accessible relevant knowledge is often not applied to the task (Weber, 2001), (b) less accessible relevant task features are often overlooked, and (c) more accessible irrelevant task features are often overemphasized (Evans, 2008).

Analysis is frequently slow, deliberate, requires much cognitive effort, and is developed through reflective and logical thinking (Evans, 2010). Analytical reasoning is often brought into action in response to an intuitive representation of a task (Evans, 2010; Kahneman, 2002). However, the power of intuition may result in analytical reasoning being (a) bound to a biased or incomplete intuitive representation or (b) invoked solely to justify an intuitive representation, thus failing to consider alternative representations of a task (Thompson, 2009). Thus, analytical reasoning may not be able to overcome faulty intuitive reasoning.

Method of Inquiry

The participants in this study were four undergraduate mathematics majors, called Ann, Brian, Chris, and Dave, in transition-to-proof mathematics courses at two private liberal arts colleges in Ohio and West Virginia and one graduate student in mathematics, called Ben, at a public university in Ohio. I conducted individual, semistructured, task-based interviews with each participant (Goldin, 2000). Each interview was audio-recorded and transcribed. During the interview, participants worked on three prove-or-disprove tasks, including the following:

> *Monotonicity task*: Definitions: A function f: $\mathbb{R} \rightarrow \mathbb{R}$ is said to be **increasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) < f(x_2)$). Similarly, a function f: $\mathbb{R} \rightarrow \mathbb{R}$ is said to be **decreasing** if and only if for all $x_1, x_2 \in \mathbb{R}$, ($x_1 < x_2$ implies $f(x_1) > f(x_2)$). Prove or disprove: If f: $\mathbb{R} \rightarrow \mathbb{R}$ and g: $\mathbb{R} \rightarrow \mathbb{R}$ are decreasing on an interval I, then the composite function $f \circ g$ is increasing on I.

I instructed the students to think aloud during the tasks and to clarify or expand on their thinking as necessary. Upon completion of the tasks, I asked the students about difficulties they had with the tasks and general strategies they used for prove-or-disprove tasks.

Analysis included the following: (a) categorizing students' reasoning as intuitive or analytical, (b) identifying errors in students' reasoning, (c) classifying errors as intuitive or analytical, (d) categorizing intuitive errors as relevance or attribute substitution errors, (e) determining the impact of intuitive errors on analytical reasoning.

Preliminary Results

Each student believed that the false statement in the monotonicity task was true and attempted to prove it. Ann and Brian each committed an attribute substitution error, and Chris, Dave, and Ben each committed a relevance error.

Ann and Brian made attribute substitution errors that prohibited them from making significant progress on the task. Ann substituted the similar concept of *negative times negative equals positive* for the task concept of *decreasing composed with decreasing equals increasing*. Brian substituted the incorrect concept *odd times odd equals even* in place of *decreasing composed with decreasing equals increasing*. As an example of attribute substitution, consider Ann's intuitive response to this task:

> Well my first thought is just simply that if the two functions f and g are both decreasing, then at that point, then both of the slopes would have to be negative, or something in there would have to be negative, which, and then I go to the simple [idea] that a negative times a negative is a positive. That would be increasing.

Ann changed the given task, replacing it with a similar and more accessible task. She then illustrated her idea with an example in which she composed two negative functions resulting in a positive function. Thus, her analytical reasoning supported her intuition. However, she was unable to begin a proof of the task statement.

Chris, Dave, and Ben made relevance errors that led them to construct false proofs for the task. They each ignored the interval restriction in the task, responding as if the task was to prove or disprove the following: If f: $\mathbb{R} \rightarrow \mathbb{R}$ and g: $\mathbb{R} \rightarrow \mathbb{R}$ are decreasing, then the composite function $f \circ g$ is increasing. The proofs that Chris, Dave, and Ben constructed were correct proofs for this similar, more accessible task. Each student produced essentially the same false proof of the task, for which Ben's proof is representative:

> Suppose $x_1 < x_2$. Then g decreasing implies $g(x_1) > g(x_2)$. Now apply f to $g(x_1)$ and $g(x_2)$. $g(x_2) < g(x_1) \rightarrow f(g(x_2)) > f(g(x_1))$. Started with $x_1 < x_2$, conclude $f(g(x_1)) < f(g(x_2))$. Therefore $f \circ g$ is increasing.

Each student focused on the basic idea that a decreasing function composed with a decreasing function would result in an increasing function. There was no spoken or written consideration of the interval restriction by any of these three students.

Discussion

Each student demonstrated some intuition on the monotonicity task and created a deficient intuitive representation of the task based on a systematic intuitive error. This led each student incorrectly to judge the statement to be true. The students' subsequent work was based on this ill-formed intuitive representation, but their analytical reasoning was mathematically correct. Ann and Brian constructed examples to help them support their intuitive representations and correctly interpreted the information in the examples, but were unable to move beyond examples to more general representations of the task. Chris, Dave, and Ben constructed correct proofs to a similar, yet different, mathematical assertion.

These students demonstrated intuition, valid mathematical reasoning, and use of relevant mathematical knowledge on this task. Furthermore, these students connected their intuition to their analytical work on the task. So why did they still all think this false statement was true and back up their claim with legitimate mathematical work? These students were victims of the quick, automatic processing of their intuition which developed an inaccurate or incomplete representation of the task for their analytical reasoning to process. Due to the power of these intuitive errors, the students employed analysis to support them rather than correct them. Thus, the students' intuition had already warped the task before their analytical reasoning had the chance to respond.

Questions for the Audience

Does the dual-process theory perspective have useful practical implications? Is the monotonicity task a “trick” question? Would a larger scale comparison study with undergraduate and graduate students be a worthwhile step to further this work?

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