THE MERITS OF COLLABORATION BETWEEN MATHEMATICIANS AND MATHEMATICS EDUCATORS ON THE DESIGN AND IMPLEMENTATION OF AN UNDERGRADUATE COURSE ON MATHEMATICAL PROOF AND PROVING

Pooneh Sabouri, Michael Thoms, Orit Zaslavsky
New York University

The goal of our study was to characterize the processes and to identify the ways in which different kinds of expertise (mathematics vs. mathematics education) unfolded in the planning and teaching of an undergraduate course on Mathematical Proof and Proving (MPP), which was co-taught by a professor of mathematics and a professor of mathematics education. The content of the course consisted of topics that were supposed to be familiar to the students, i.e., high school level algebra, geometry, and basic number theory. In particular, we looked at a case study describing the design and implementation of a particular task in order to help understand an instance in which each professor's expertise contributed to the course and complemented the other. The findings indicate that by co-teaching and constantly reflecting on their thinking and teaching, the instructors became aware of the added value of working together and the unique contribution each one had.

Key words: Mathematical Proof and Proving; Undergraduate Course; Problem-based Learning; Community of Practice

Setting the Stage: Context and goals

The focus of our study is on the design and implementation of a special undergraduate course on Mathematical Proof and Proving (MPP)[1]. Two common assumptions led to initiating this course: 1. Mathematical proof and proving are at the heart of mathematics; and 2. The notion of formal proof and the activity of mathematically proving are dauntingly difficult even for most good undergraduate students (Harel and Sowder, 2007).

The notion of proof is often incorporated into other mathematical courses and typically does not constitute the focal topic of one particular undergraduate course. There are many transitional courses in mathematics, most of which combine learning about proof with learning fundamental unfamiliar topics in mathematics. Consequently, the cognitive load on students is high and they encounter more difficulty than necessary since they need to deal with too many things at the same time: advanced mathematical ideas as well as proof and proving. The intention of the MPP course was to build on students' existing mathematical knowledge, and to draw on learning activities that involve familiar topics such as high school level algebra and geometry, and basic number theory (e.g., familiar properties of integers such as divisibility).

The challenge of attending to students' learning difficulties and at the same time maintaining an appropriate level of sound mathematics led to a collaboration between mathematicians and mathematics educators. Moreover, the MPP course was designed and co-taught by two instructors – a full professor of mathematics and a full professor of mathematics education. The initial goal of this collaboration was for the mathematics educator to bring her expertise on teaching mathematics (in general, and of proof and proving, in particular) and on students' difficulties in learning to prove, and for the mathematician to bring his expertise in the discipline of mathematics and the knowledge and understanding of MPP that students' need for successful participation in more advanced

[1] This paper is based on work supported by the National Science Foundation under Grant No. 1044809.

undergraduate mathematics content courses. This collaboration stemmed from a mutual respect for each other's role and potential contribution, and the recognition that there is much to learn from each other. In reality, there were additional mathematicians and mathematics educators involved in various stages of this process.

While sharing the same concerns and long-term goals for the course, each instructor brought a different perspective on how students should be learning MPP and how to attend to their difficulties. From the outset it became clear that although the structure and syllabus of the course were pre-determined in full agreement between the two instructors, each instructor has his/her own views and interpretations, and that the joint efforts to produce an MPP course that would address the above concerns would require an ongoing professional dialogue and reflection. The big challenge was to bridge between the different perspectives and use these differences as a springboard to enhance the course.

The goal of our study was to characterize the processes and to identify the ways in which the different kinds of expertise (mathematics vs. mathematics education) unfolded in the actual planning and teaching of the MPP course; in particular, we looked for instances that would help understand how each expertise contributed to the course and complemented the other.

Conceptual Framework

Our study stems from two theoretical perspectives. One supported the design of the course. The other supported our approach to the design and study of the collaborative work between the two communities represented by the two instructors.

The following perspectives on learning and teaching guided the design of the MPP course: 1. Students' interactions and classroom discourse contribute to learning [to prove] (Yackel, Rasmussen, and King, 2000; Zaslavsky and Shir, 2005; Smith, Nicholas, Yoo, and Oehler, 2009); 2. Tasks play a significant role in learning (Henningsen and Stein, 1997); 3. Uncertainty promotes the need to prove (Dewey, 1933; Fischbein, 1987; Harel and Sowder, 2007, 2009; Zaslavsky, 2005; Zaslavsky et al, 2011); 4. Class discussions and activities should address students' anticipated/manifested preconceptions and difficulties (Harel and Sowder, 2007, 2009; Weber, 2001; Reid, 2002; Buchbinder and Zaslavsky, 2009).

The decision to design and co-teach the course collaboratively, assigning two full professors as the MPP course instructors, is in a way a response to issues raised by Harel and Sowder (2009). Their study indicates that while mathematicians who teach undergraduate courses in mathematics have a broad and deep mathematical knowledge/understanding, many are not necessarily fully aware of students' difficulties in learning to prove, or of effective ways to scaffold their learning. In our work, the team of mathematics educators and mathematicians is viewed as a community of practice. The team consisted of 3 mathematics educators – one full professor (Olga) and two doctoral students (Mark and Pola), and 3 mathematicians – two full professors (Jim and Frank) and one doctoral student (Sam). All the names in this paper are pseudonyms. Olga and Jim were the instructors of the course. Mark and Sam served as teaching assistants (TAs), Pola served as research assistant on the evaluation staff, and Frank was involved primarily in the planning sessions. The members varied with respect to their expertise and experience, as well as their roles, which is one of the characteristics that Roth (1998) considers essential to a community. Theories of communities of practices provide us with tools for analyzing the various kinds of learning of the members of the community as well as the contribution of each member to the shared goals of the community (Rogoff, 1990; Roth, 1998; Lave & Wenger, 1991). These theories consider knowledge as developing socially within communities of practice.

An integral characteristic of our community of practice is associated with the notion of reflective practice (Dewey, 1933; Schön, 1983). The notions of reflection on-action and reflection in-action have been recognized as effective components that can contribute to the growth of teachers' knowledge about their practice. In our study, reflection was a key issue for the development of the instructors' awareness and understandings related to teaching and learning to prove.

Data Sources and Analysis

The data for this study consisted of video-tapes and field notes of all the classes in the semester (13), audio-records and field notes of weekly meetings held a day after each class, and email conversations between the team members. In addition, students' written homework and TA's comments and grades were scanned and documented.

The methodology employed in the study followed a qualitative research paradigm in which the researcher is part of the community under investigation. It borrows from Strauss and Corbin's (1998) Grounded Theory, according to which the researcher's perspective crystallizes as the evidence, documents, and pieces of information accumulate in an inductive process from which a theory emerges. The researcher acts as a reflective practitioner (Schön, 1983) whose ongoing reflectiveness and interpretativeness are essential components (Erickson, 1986). In our case, the researchers were members of the community of practice that they investigated.

In the following section, we present an illustrative case that portrays the kinds of negotiations between and mutual contributions of the two instructors during a sequence in which they first discussed a particular task in a pre-class planning meeting, then implemented it in the classroom, and finally reflected on its implementation in a post-class meeting. The focus of this study is not on student learning, but rather on describing the mutual learning that occurred around this task by members the community. Through this case, we provide a glimpse at the ways in which two instructors reflect on their mutual understanding of how to teach proof, and work towards developing a shared understanding.

An Illustrative Case

The Cyclic Task

Based on the design principles of the MPP course, the following task was posed to students during the 5th week of the course as part of a sub-unit on direct proof. Olga started by choosing a 3-digit number: 814, which is divisible by 37. She then asked the students to check whether the numbers 148 and 481 (that are obtained by a cyclic change of 814) were also divisible by 37. Much to their surprise, they found that both 148 and 481 are also divisible by 37. Then she asked them to choose another 3-digit number that is divisible by 37 and to check whether any cyclic change of digits (i.e., the first permutation) is also divisible by 37. This way, students jointly tried out several cases that satisfied this property. At this point, Olga asked them: "*is this a coincidence?*" Although it seemed to work for the examples they chose, they were uncertain whether it would always work. Thus, this question led to the formulation of a conjecture and an attempt to prove or disprove it. The conjecture that was formulated was: "*If a 3-digit number is divisible by 37, then any 3-digit number that is obtained by a cyclic change of order of digits is also divisible by 37.*"

This task was selected for two main reasons. First, minimal knowledge about number theory and divisibility is required to form a conjecture, and both professors anticipated that the construction of a valid proof would be within students' abilities. Although the mathematical content of the cyclic task has its roots in elementary number theory, and can be explained by appealing to permutation theory and modular arithmetic, it is possible for students to construct a proof by appealing to the nature of the decimal system notation and the grouping of like-terms. For example, it is possible to show the implication of one cycle by expressing the number 'xyz' as $100x + 10y + z$, expressing 'yzx' in a similar form, and use substitution and grouping of like terms to show that if 'xyz' is divisible by 37, then 'yzx' can be represented as a sum of terms that are each divisible by 37. Second, the proof of the statement is not trivial. Tasks that evoke feelings of uncertainty in students have the potential to support meaningful learning situations (Zaslavsky, 2005; Zaslavsky et al, 2011), as well as to create a need for certainty that Harel & Sowder (2009) describe as one of the five elements that constitute an intellectual need particularly relevant to learning mathematical proof. Another affordance of this task is that it can be extended by asking students to reflect on their proof and determine if there are other numbers for which the cyclic pattern holds.

Pre-Class Planning of the Cyclic Task

During the pre-class instructors' meeting, Olga and Jim decided that they would give students substantial time to work on the cyclic task in class, corresponding to their goal of making the MPP course a problem-based course. Olga suggested that they give students one hour, as from her prior experiences giving the task to students she believed that discovering a pattern and then trying to prove it would not be easy. Jim, on the other hand, was hesitant about giving students this much time to work on their own: "*I am absolutely convinced the majority will not get it… it will be a very frustrating hour*." Olga convinced him that class time spent on the task would be interspersed with full class discussion and sharing, and it would not be the case that students would work on it for an hour in isolation.

Jim suggested that before proceeding to the cyclic task Olga review a problem from their previous homework on the "divisibility-by-3" rule. This homework was deliberately assigned to them a week earlier in order to prepare them for the cyclic task. They were asked to prove that: *If the sum of the digits of an integer n is divisible by 3, then n is divisible by 3*, for 4-digit integers.

Both Olga and Jim agreed that this was a good idea due to the conceptual similarity between this homework assignment and the cyclic task. They disagreed only in how to make the transition between the two. Jim suggested that after reviewing the "divisibility-by-3" rule "*we give this (cyclic) as an exercise and say can you use some of these ideas (from the homework) to show this*?" Olga argued against making this connection explicit. By the end of this pre-class planning meeting, each professor assumed that the other saw the merits of the task in a manner similar to their own.

The Unfolding the Cyclic Task in Class

As planned, during the first half hour of the lesson, Olga worked with students to construct a proof of the "divisibility-by-3" rule, as none of them had successively completed this homework assignment. Students learned how to represent a 3-digit number as a sum of powers of ten, and the affordances in the proof construction of regrouping this representation (i.e. $100x + 10y + z = (99x + x) + (9y + y) + z = (99x + 9y) + (x + y + z)$.

Olga introduced the cyclic task slowly, by having students verify that the claim was in fact accurate with two different examples. Students were instructed to work on the task in groups. During the group work, Jim sat with one group and listened to their discussion, while Olga walked around to check on different groups. After ten minutes, one group indicated that they made some progress and Olga asked them to share their work on the board. The group had discovered that the sum of the three cyclic changes of a general three-digit number was always divisible by 37, but did not know what to do with this information (Figure 1).

Jim raised a concern, asking one of the students – David, if his observation was based on the given statement (i.e. that $100x + 10y + z$ is divisible by 37), and when David replied that it was not, Jim noted: *"It could be any number, [this] line of thinking is going to tell you that oh, every number is divisible by thirty-seven."*

To get students back on track, Jim wrote on the board what needs to be proved (RTP). For this he used the cyclic change occurring in a counter-clockwise direction (Figure 2.a.), which was different from the clockwise direction in which the problem had been framed. Some students seemed puzzled by this choice. In response to a student's question of whether it was necessary to prove each of the two cyclic changes ($xyz \rightarrow yzx \rightarrow zxy$), Jim answered: *"If you prove one of them, you prove both of them."* At this point, Olga interrupted Jim and changed the notation on the board, to reduce students' confusion (Figure 2.b). In the post-class discussion, she explained her move.

Post Class Reflection

During the post-class meeting, Olga raised two issues about the lesson:

"[I] believe if you knew my thinking, you probably would have not jumped at some points. I suddenly realized, oh you are not aware why I am doing it".

Jim agreed: *"we should probably talk about these things before we go to the classroom to know where we are headed…"*

She explained that to reduce the complexity of the problem she wanted everyone to use at first the clockwise cyclic change "*because those [students] who started clockwise started using this notation, so I don't want to make it more complicated for people".* For Jim it made no difference whether to choose a clockwise or a counter-clockwise cyclic change. Clearly, from a mathematical point of view this is right. He was not aware of the pedagogical value of making this distinction.

Moreover, Olga planned to allow the students to prove that the first cyclic change is divisible by 37 and then move to the second one, in accordance with the "repeated reasoning" principle of the DNR (Harel and Sowder, 2009). Only after they did the proof again, for the second case, would she pose the question of whether the second proof was needed: *"You [Jim] would not even ask that question, since it is so obvious [to you]… I thought you raised this issue before they were prepared for that... Clearly you don't need to do it again, because it is like, the big idea of without the loss of generality.... and these are issues that are deep and I wanted them to think about them."*

It should be noted that in the pre and post-class meetings the entire team (of 6) participated. For brevity, we did not bring the full scope of the conversations surrounding the Cyclic Task.

Concluding Remarks: What is this a case of?

The conversation between Olga and Jim through planning and reflecting on this lesson is an example of how members of this unique community of practice discussed, negotiated meaning and came up with shared understandings and better informed ideas of each other's perspective about the practice of teaching proof to students.

In the pre-class meeting, neither Jim nor Olga described how the problem would unfold, as they took it as shared knowledge. In the actual implementation of the lesson, they both realized that although there is always room for spontaneous moves, they should be aware of each other's thinking. Nonetheless, each of them contributed to the lesson without prior coordination with the other. In some ways they complemented each other in a supporting way (e.g., the way Jim commented on the work that David presented (Figure 1), namely, his group's observation that the sum of any 3-digit integer and its two cyclic permutations is always divisible by 37). None of them anticipated this observation, which required spontaneous action. However, Jim's next spontaneous action, using the counter-clockwise cyclic change, before they were prepared for it, did not concur with Olga's pre-planned trajectory that aimed at addressing students' anticipated difficulties more gradually.

The illustrative case that we presented above captures ways in which the collaboration between these two experts made them conscious of each other's considerations and of the importance of questioning their assumptions and negotiating them. More generally, this study seized the opportunity to develop a community of practice that did not exist before, and to trace the process of exchanging expertise and learning from one another partly by apprenticeship and partly by reflecting in and on action and negotiating meaning.

Figures

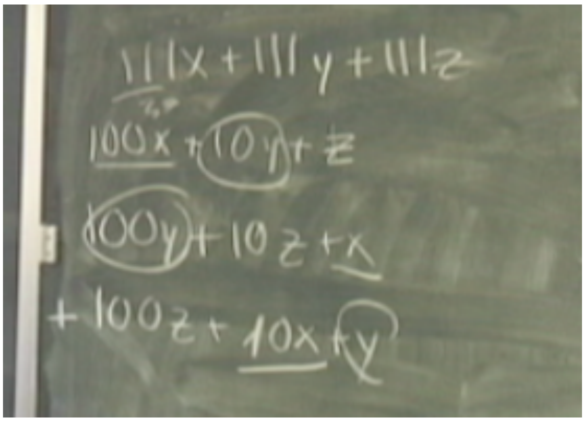

Figure 1. David's group work

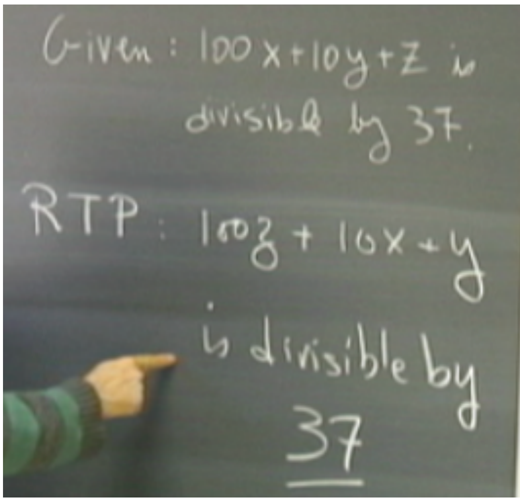

Figure 2.a. Jim's initial presentation - a counter-clockwise cyclic change

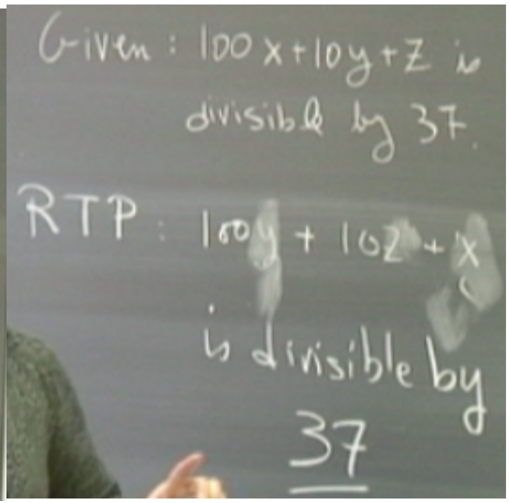

Figure 2.b. Olga's modified notation - a clockwise cyclic change

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