TWO STUDENTS' INTERPRETATION OF RATE OF CHANGE IN SPACE

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This paper describes a model of the understandings of two first-semester calculus students, Brian and Neil, as they participated in a teaching experiment focused on exploring ways of thinking about rate of change of two-variable functions. I describe the students' construction of directional derivative as they attempted to generalize their understanding of one-variable rate of change functions, and characterize the importance of quantitative and covariational reasoning in this generalization.

Key words: Rate of change, covariation, functions, graphs, quantitative reasoning.

Background and Research Question

Mathematics and science are concerned with characterizing the behavior of complicated systems. The transition students make as they shift from thinking about systems with two quantities varying to systems with three or more quantities varying has not been fully explored. In response to the need to understand how students model change in complicated systems, this study sought to gain insight into,

> *What ways of thinking do students reveal in a teaching experiment that is focused on the meaning and measurement of rate of change in space?*

Understanding rate of change is foundational to ways of thinking about ideas in calculus, yet many students possess difficulties reasoning about rate (Carlson et al, 2001; Carlson et al., 2003; Monk, 1987; Rasmussen, 2000; Thompson & Silverman, 2008). Students' difficulties understanding rate of change range from problems interpreting the derivative on a graph (Asiala et al., 1997) and focusing on cosmetic features of a graph (Ellis, 2009). Thompson (1994) found that the difficulties student's displayed in understanding the fundamental theorem arose from impoverished concepts of rate of change and incoherent images of functional covariation. Thompson described a coherent way of thinking about average rate of change of a quantity as, "if a quantity were to grow in measure at a constant rate of change with respect to a uniformly changing quantity, then we would end up with the same amount of change in the dependent quantity as actually occurred". The characterizations of difficulties students have in thinking about rate and Thompson's scheme for thinking about rate were the foundation for a conceptual analysis. The following conceptual analysis uses Thompson's scheme of meanings and extends it to rate in space.

Theoretical Framework and Conceptual Analysis

I based the study on quantitative and covariational reasoning, which served as the theoretical lens through which I constructed the tasks. The following conceptual analysis represents a plausible way for a student to coherently understand rate of change in space, and served as the basis for the construction of tasks.

Instantaneous rate of change can be thought of as an average rate of change over an infinitesimal interval. The average rate of change of a quantity C (f(x,y)) with respect to quantities A (x) and quantity B (y) in a given direction in space can be thought of as the constant rate at which another quantity D would need to change with respect to quantities A and B to produce the same change as quantity C in the same direction that (x,y) changed. Then quantity D accrues in a constant proportional relationship with quantity A, and simultaneously accrues in a proportional relationship with quantity B.

The average rate of change between two points in space is the constant rate at which another function g(x,y) would need to change with respect to x and y over the intervals $[x_0,x_1]$ and $[y_0,y_1]$ to produce the same change as f(x,y) over those intervals. The function g(x,y) must change at a constant rate with respect to x and a constant rate with respect to y and those constant rates must remain in an invariant proportion. An "exact" rate of change is an average rate of change of f(x,y) over an infinitesimal interval of $[x_0,x_1]$ and$[y_0,y_1]$.

Thinking about the rate of change of f(x,y) as above supports thinking that any accrual d of either x and y must be made in constant proportion b/a. This proportion a/b actually specifies the direction of change. Thus, rate of change of f(x,y) with respect to x can be reformulated as $f_u'(x,y) = \lim_{d \to 0}[f(x+h, y+k) - f(x,y)]/d$, where $ad = h$ and $bd = k$ so h must be a/b'ths of k and k must be b/a'ths of h. Then, d can be thought of in the same way as h in the one-variable case, where the derivative is an average rate of change of a function over infinitesimal intervals and the proportional correspondence between h and k means they have a linear relationship resulting in approaching the point (x_0,y_0) along a line. This conceptual analysis served as the basis for the construction of tasks, and interpretation of student responses during the teaching experiment.

Method

This study used Steffe and Thompson's (2000) account of a teaching experiment to build models of students' ways of thinking about mathematical ideas by focusing on the mathematics of students, which refers to ways of thinking that, were a student to have them, would make the student's words and actions sensible for the student. I generated a set of hypotheses about the students' ways of thinking, and used the tasks to test these hypotheses using grounded theory and open and axial coding. The cycle of task hypotheses testing and generation is depicted below (Figure 1).

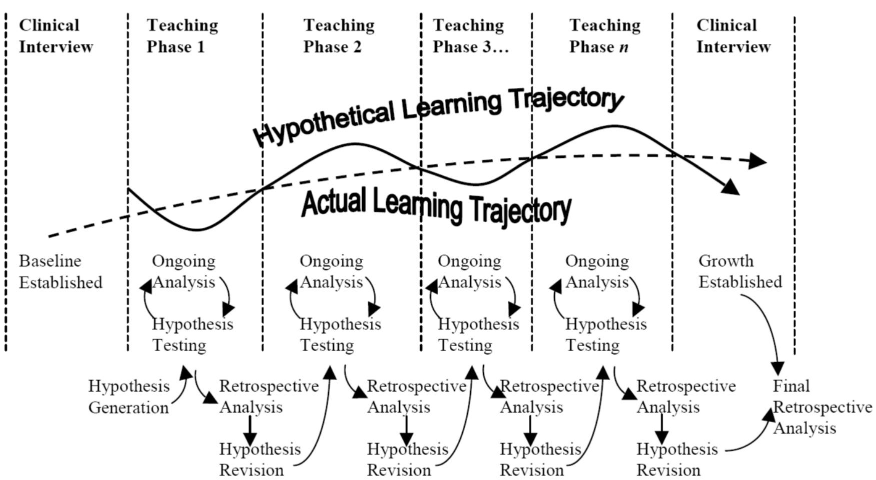


Figure 1. Characterzation of a teaching experiment methodology.

Results & Discussion

The following excerpt centered on how Neil and Brian would interpret the meaning of rate of change at a point in space.

Excerpt 1

EW: How would you think about rate of change at a point in space?

Brian: My first though is that it has multiple rates of change, kind of like sitting on a hill, depending where you look, the steepness, slope at that point can be different.

Neil: I agree with that, I thought about the kind of example too, or just sitting on the surface we swept out, maybe we can use the z-x and z-y rates of change?

Brian: Yeah! Umm, let's see though, if we want to make a rate of change function and then graph it in space, we need to figure out a way to program it, and do a sweeping out.

Neil: What about just plugging in *x*, *y* and the rate of change? Oh, I guess we don't know the rate of change yet, so my point thing wouldn't work.

Brian: Alright, for z-x, it is kind of like, okay let's back up, let's say we are at a point (a,b,c) in space. Then for z-x, we fix *y* at *b*, then do the normal rate of change except it has to be two-variables.

Neil: Yeah, that makes sense, so we need an *h*, maybe like *f*(x+h,b)-*f*(x,y), then divided by *h*, and then for z-y, we just say y+h and fix *x* at *a*?

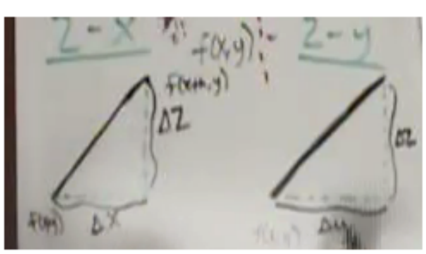
Brian: Yeah, let's go with that.

EW: Okay, so where do you want to go next then, what is your plan?

Brian: We need the two rates of change to make an overall rate of change function, then we can graph it by doing the sweeping out I think.

Neil: I'd rather just draw the two calculus triangles that I am imagining each in a perspective.

Brian's description of rate of change (Excerpt 1, lines 2-3), indicated he was thinking about rate of change in a direction "at" a point on the surface of the function's graph. Brian's suggestion of considering multiple rates of change from a perspective (Excerpt 1, lines 14-16) led to Neil's sketching of perspective dependent calculus triangles (Figure 2). Their descriptions of multiple rates of change, as well as specific rate of change functions for z-x and z-y perspectives, indicated they were imagining rate of change occurring in at least two directions. I intended to understand if they imagined the rates of change occurring simultaneously.



$$r_{f_x}(x,y) = \frac{f(x+h_1, y) - f(x,y)}{h_1} \quad (Z\text{-}x)$$

$$r_{f_y}(x,y) = \frac{f(x, y+h_2) - f(x,y)}{h_2} \quad (z\text{-}y)$$

Figure 2. Calculus triangles from the z-x and z-y perspectives.

Brian described the z-x perspective rate of change as the average rate of change of *f* with respect to *x* while holding *y* constant, and z-y as the average rate of change of *f* with respect to *y* while holding *x* constant. In the following excerpt, I asked Neil and Brian to expand on their description of their perspective dependent calculus triangles, in particular their use of h_1 and h_2 in the denominators of the open form rate of change functions. I anticipated that thinking about the relationship of h_1 and h_2 would be critical to their creating a need for considering rate of change in a direction.

Excerpt 2

EW: So I noticed that you guys constructed your calculus triangles in each perspective and labeled the *h*'s as different. Can you say more?

Neil: Yeah, well basically the *h*'s are independent, so they don't have to be the same, but I guess they could be.

Brian: I was thinking about this more last night, and had a sort of moment. We talked earlier about being on a hill, or walking on a function, for example, and the rate of change depended on the direction you were facing, I think the same thing applies here.

EW: What do you mean by a direction?

Brian: Okay, so let's say we head from a point, or we went directly Northeast, and imagine we are doing this from overhead, it was looking at the x-y perspective that made me think of this.

EW: Okay, so if you were heading Northeast, what's the significance of that?

Brian: Then we know how the two *h*'s are related to each other, because most of the time you don't go in a direction of just north, west, east, whatever, you head in some combination.

Neil: Okay, not sure I am following, but basically if we head Northeast, the change in *x* and change in *y* would be equal, a different direction, change in *x* could be twice as big as change in *y*, which are like the *h*'s right?

Brian: Yeah, let me make an illustration here. Then we can just call the numerator a change in *z*, $f(x+h_1, y+h_2) - f(x,y)$ either *h* value that we want to.

Brian introduced direction as a way to account for all possible rates of change in space (Excerpt 2, lines 10-12). I believed Brian had an image of the relationship between h_1 and h_2 to define a direction (Excerpt 2, lines 15-17). Brian's key insight was that *any* direction was more general than considering only z-x and z-y perspectives.

Neil and Brian continued to work on developing a two-variable open form rate of change function, and they agreed that the numerator represented a change in the output, represented by $f(x+h_1, y+h_2) - f(x,y)$, where either h_1 or h_2 was written in terms of the other *h*-value. However, both Neil and Brian questioned how they have a single denominator that represented a change in *x* and a change in *y*. Even though they saw that h_1 and h_2 depended on each other, that dependence did not immediately resolve their issue of what change to represent in the denominator. I believed that this was because they were focused on trying to represent a single change in the denominator while they understood that there were changes in both *x* and *y*.

Excerpt 3

EW: So, what is your denominator in the function, your conjecture?
Brian: I was thinking either *h*-value, whichever you have in the numerator.
Neil: But doesn't that kind of just delete it, it goes away?
Brian: No, you just define it in terms of h_1, if you are talking about h_2.
EW: So if one *h* is in the denominator, and we make that value small, what happens to the other *h* value??
Brian: Well, oh yeah, it becomes small as well, it's not like the *h*'s have the same value, but they can end up getting so small it doesn't matter.
Neil: Ah, I see what you mean. So like in class we talk about making changes so small, because we have an equation to relate the two *h*'s, then when one gets really small, the other one has to get smaller too? I was thinking multiply in the denominator I guess, but that doesn't need to happen, you just need one of them to become small because they are related, the *h*'s I mean, so then both do.
Brian: So, I guess that's our overall rate function for two variables. Now we can sort of program the points to do the graph.

Brian's insight that the changes in *x* and *y* became smaller in tandem allowed him to conjecture that using a single parameter in the denominator (Excerpt 3, lines 7-8) was acceptable (Excerpt 3, line 2). Neil appeared to focus on deleting one of the parameters, but Brian's insight allowed him to think about the equation they had specified between h_1 and h_2. By imagining progressively smaller values for h_1, he found that h_2 became smaller as well given the proportional relationship specified by choosing a direction in space (see Figure 3). These insights allowed them to construct an average rate of change function (Figure 3).

$$r_f(x,y) = f(x+h_1, y+h_1 d) - f(x,y) \,/\, h_1$$

direction = d d= fixed $\frac{\Delta y}{\Delta x}$ $h_1 = \Delta x$ $h_2 = \Delta y$

Figure 3. Brian and Neil's two variable average rate of change function.

Conclusions

Brian and Neil problematized how to interpret and calculate a rate of change at a point in space because they were concerned with defining rate of change at a point in space. Neil's construction of "simultaneous calculus triangles" and Brian's reflection on them as a way to represent partial rates of change indicated to me that they were imagining change occurring in different directions in space. The x-y perspective allowed them to think about relating the *simultaneous* changes in *x* and *y* by considering a direction, and supported their construction of an open form rate of change function. By attempting to generate the graph of the rate of change function, Brian, and then Neil, were able to problematize the existence of rate of change at a point in space. This problematization occurred without formal focus on directional derivative, and was based off of the students' understanding of a one-variable rate of change function.

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