UNDERSTANDING ABTRACT ALGEBRA CONCEPTS

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ABSTRACT: *This study discusses various theoretical perspectives on abstract concept formation. Students' reasoning about abstract objects is described based on proposition that abstraction is a shift from abstract to concrete. Existing literature suggested a theoretical framework for the study. The framework describes process of abstraction through its elements: assembling, theoretical generalization into abstract entity, and articulation. The elements of the theoretical framework are identified from students' interpretations of and manipulations with elementary abstract algebra concepts including the concepts of binary operation, identity and inverse element, group, subgroup.*
To accomplish this, students participating in the abstract algebra class were observed during one semester. Analysis of interviews and written artifacts revealed different aspects of students' reasoning about abstract objects. Discussion of the analysis allowed formulating characteristics of processes of abstraction and generalization. The study offers theoretical assumptions on students reasoning about abstract objects. The assumptions, therefore, provide implications for instructions and future research.

KEYWORDS: Abstraction, Generalization, Abstract Algebra, Group Theory

Introduction. Abstract thought is considered to be the highest accomplishment of the human intellect as well as its most powerful tool (Ohlsson, Lehitinen, 1997). Even though some mathematical problems can be solved by guessing, trial and error, or experimenting (Halmos, 1982), there is still a need for abstract thought. There is support (Ferguson, 1986) for the hypothesis that abstraction anxiety is an important factor of mathematics anxiety, especially concerning topics which are introduced in the middle grades. By understanding an abstract concept formation we will be able to help students to overcome this anxiety.

This paper presents results of the exploration of the process of abstraction and gives a description of its components and outcomes. The goal is to understand the nature and acquisition of abstraction, so we can help students to bridge the gap from the abstract to concrete. Qualitative approach has been used to reach the goal. Analysis of students' concept formation (knowledge of abstract/mathematical object) is consistent with the tradition of a grounded theory (Charmaz, 2003; Glaser & Strauss, 1967). The study was conducted in the content of group theory.

Theoretical Framework. Piaget (1970a, 1970b) considers two types of cognition: association and assimilation, stating that assimilation implies integration of structures. Piaget distinguishes three aspects of the process of assimilation: repetition, recognition and generalization, which can closely follow each other. In his papers about advanced mathematical thinking, Dubinsky (1991a, 1991) proposes that the concept of reflective abstraction, introduced by Piaget, can be a powerful tool in the process of investigating mathematical thinking and advanced thinking in particular.

In the late 1980s Ed Dubinsky and his colleagues (Clark at al., 1997) started to develop a theory that describes what can possible be going on in the mind of an individual when he or she is attempting to learn a mathematical concept. In recent years, the mathematics education community at large started to work on developing a theoretical framework and a curriculum for undergraduate mathematics education. Asiala (Asiala et al, 1996) reported the results on their work: based on the theories of cognitive construction developed by Piaget for younger children, Dubinsky and his colleagues proposed the APOS (action – process – object – schema) theory. A number of studies on topics from calculus and abstract algebra (Zazkis & Dubinsky, 1996; Dubinsky at al, 1994; Brown at al, 1997; etc) were reported using this framework.

The theoretical approach, described by Davydov (1972/1990), is highly relevant to educational research and practice. His theory seems incompatible with the classical Aristotelian theory, in which abstraction is considered to be a mental shift from concrete objects to its mental representation – abstract objects. By contrast, for Davydov, as well as for Ohlsson, Lehitinen (1997), Mitchelmore and White (1994, 1999), Harel and Tall (1991, 1995), abstraction is a shift from abstract to concrete. Ohlsson and Lehitinen provide us with historical examples of scientific theories development; Davydov also gives historical examples and, at the same time criticizes the empirical view on instruction by claiming that empirical character of generalization may cause difficulties in students' mathematical understanding.

Following Piaget (1970 a), the framework for this study considers the process of abstraction as a derivation of higher-order structures form the previously acquired lower-order structures. Moreover, the two types of abstraction are distinguished. One of these types is simple or empirical abstraction – from concrete instances to abstract idea. The second type then is more isolated from the concrete. Davydov (1972/1990) calls this type of abstraction "theoretical abstraction". Theoretical abstraction, based on Davydov's theory, is the theoretical analysis of objects (concrete or previously abstracted) and the construction of a system that summarizes the previous knowledge into the new concept (mathematical object), so it is ready to be applied to particular objects. This abstraction appears from abstract toward concrete and its function is the object's recognition. According to present research, the second type of abstraction is commonly accepted as essential in the process of learning deep mathematical ideas. Similarly, there are two types of generalization – generalization in a sense of Ohlsson and Lehitinen perspectives (which coincides with empirical perspective, described by Davydov); and theoretical generalization. Theoretical generalization is the process of identifying deep, structural similarities, which in turn, identify the inner connections with previously learned ideas. The process of theoretical abstraction leads us to the creation of a new mental object, while the process of theoretical generalization extends the meaning of this new object, searching for inner connections and connections with other structures.

In summary, the genesis of new abstract idea looks like following: (0) initial abstractions; (1) grouping previously acquired abstractions (initial abstractions in a very elementary level); (2) generalization to identify inner connections with previously learned ideas; (3) the shift from abstract idea to a particular example to articulate a new concept. Note that at some level of cognitive development initial abstractions become obsolete since sufficient more complex and concrete-independent ideas are already acquired. The result of this genesis is a new structure which is more complex and more abstract compared to the assembled ideas. Hence, we have hierarchical construction of knowledge, where the next idea is more advanced than the previous one. Moreover, cognitive function of abstraction (from now on, abstraction and generalization are theoretical abstraction and generalization, as defined above) is to enable the assembly of previously existed ideas into a more complex structure. The main function of abstraction is recognition of the object as belonging to a certain class; while construction of a certain class is the main function of generalization, which is making connections between objects (see Fig 1). The framework suggests the design of the study and helps to ground the methodology and data collection.

Methodology. To answer the questions above, 22 students, participating in undergraduate Abstract Algebra course were observed during class periods during one semester. Written assignments (quizzes, homework, exams) were collected from all participants. A group of participants (7 students) was interviewed three times during the semester.

Research Questions. The following questions were formulated based on the theoretical framework:

- What notions and ideas do students use when they recognize a mathematical object, and why? (what are students using: definitions, properties, visualization, previously learned constructs, or something else?)
- What are the characteristics of students' mathematical knowledge acquisition in the transition from more concrete to more theoretical problem solving activity?

Discussion. The data analysis revealed a different aspect of students reasoning about abstract algebra concepts.

Understanding the concept of a binary structure

The term "binary structure" and the notation (S, ∗) normally used to represent a binary structure is usually understood by students as a mathematical object with two entrees: a set and an operation. The term and notation do not imply any necessary correspondence or relations between them. Dubinsky and colleagues (1994) discussed this problem analyzing students' understanding of groups and their subgroups. The study proposed that there are two different visions of a group: 1) a group as a set; and 2) a group as a set with an operation. Similarly for a subgroup: 1) a subgroup as a subset; and 2) a subgroup as a subset with an operation. Analysis of the data collected for this study showed related trends:

Binary Operation. Closure

The number of solutions in the data (Figure 2 and Figure 3) suggested that the students still try to assimilate the concept of a binary operation through familiar operations. Davydov (1972/1990) has proposed that the students who experience this problem try to make sense of a binary structure using empirical thoughts (empirical generalization and abstraction). Students assemble ideas of a set, its elements, an operation on any two elements, and the result of the operation on any two elements. By a simple generalization process they develop a simple abstract idea or, in other words, there is a shift from concrete operations (such that addition or multiplication, for instance) to abstract (such as operation "star" defined on set $\{a, b, c\}$).

Thus, often the process of understanding a binary operation is empirical rather than theoretical. The data provided evidence for the failure of empirical thought about binary operation during the object recognition stage. For example, when answering the following question: "Give an example of an operation on ***Z*** which has a right identity but no left identity", students often responded that division is this type of operation on ***Z*** (Figure 2). Indeed, division is not defined on ***Z*** since ***Z*** is not closed under division, and division by 0 is undefined. However, many students recognize division as a binary operation on ***Z***.

Binary Structures. Group as a set of discrete elements

Understanding a group as a structure consisting of two objects that interact with each other is complicated and novel for students. The data collected during this study suggest that some students understand a group as a set of elements. The operation in this case does not play an important role in the structure. Figure 6, for example, illustrates how students switched from one operation to another. It suggests that for students operation is not an attribute of a binary structure but rather a separate object which may be used if needed.

At the early stage of understanding the binary structure concept, students construct their knowledge based on previously learned objects. To understand a complex idea such as binary structure, students must have other ideas as parts. The process of generalization initializes connections between the elements and groups these elements in a set. Thus, the new created abstract entity simply repeats the one that already exists. In this case the operation defined on binary structure is not a part of the assembling process and exists disjointedly from the set. It follows that the abstract idea is not complete; further the main function of abstraction (object recognition) fails.

Groups and their subgroups as Binary Structures

The data showed that students often have difficulty understanding connections between a group and its subgroups, both operational and via elements. Student's responses revealed three major misconceptions about subgroups. First, for some students understanding of a subgroup is similar to the understanding of groups as sets. Interestingly, those students who at first understood a group as a set would not necessarily transfer this understanding onto subgroups and vice versa. For some students a group is a set with the operation, whereas a subgroup is just a subset, a part of a bigger structure. A subgroup exists if a subset exists. Several students claimed that the set of odd integers is a subgroup of ($\boldsymbol{Z}$, +). Second, students have problems seeing structural connections between groups and its subgroups. Sometimes they comprehend only elements connection. Students realize that a subgroup is a group itself under an assigned operation. It is not merely a subset of a bigger set; it is a structure. Nevertheless, the assigned operation is not necessarily the group operation. For example, some of the responses defended that ($\boldsymbol{Z}_n$,$+_n$) is a subgroup of ($\boldsymbol{Z}$, +), since it is a group and $\boldsymbol{Z}_n$ is a subset of $\boldsymbol{Z}$. A change of subgroups operation from the group operation to a completely different operation was also observed during problem solving activity (Figure 6). Third, some responses did not only demonstrate students' understanding of a subgroup as a subset of a given structure but, in addition, this subset is assumed to be a group itself under the given group operation. However, the concept of binary operation caused difficulty. It is illustrated by the following student's response: "the set of odd integers together with 0 is a subgroup of ($\boldsymbol{Z}$, +)".

The data also showed that students find it easy to work with concrete examples of cyclic groups. Moreover, they are very comfortable listing their subgroups and describing them. Not all the students, however, appreciate theorems which help to minimize steps in the problem solving process. The fact that students often used cyclic groups as concrete examples during problem solving suggests that cyclic groups proved themselves very useful objects for the articulation process in the group concept formation. Nevertheless, sometimes this articulation is based on empirical generalization (students observe several examples of subgroups of a cyclic group and conclude that they all must be cyclic), rather than on analysis of the inner connection within the structure. As a result, students accept the idea that if G is a group, then it is closed under the assigned operation. It follows that every nonidentity element generates a nontrivial cyclic subgroup. However, students' view of the inner connections is still not comprehensive and a group is perceived as a union of such cyclic subgroups (Figure 4).

Data analysis and theoretical perspectives suggest that when learning concepts of cyclic groups, their subgroups and cyclic subgroups, students often rely on empirical generalization since the concepts are well illustrated by a variety of concrete examples. Instead of recognizing concepts in the examples, students are looking for commonalities via empirical thought rather than theoretical.

Definitions of objects. How students use them

The theoretical framework suggests that a definition is the initial stage of concept formation. A definition suggests ideas for assembling. For example, a group is a set, closed under an assigned operation, the operation must be associative; an identity element must be in the set, and every element of the set must have an inverse. The definition puts forward some previously abstracted ideas for assembling. Analysis of the connections between the ideas, and articulation follow the assembling. Later, when concepts are being recognized in concrete problems student also must refer to definitions to collect objects from the assembling process, which must be recognized first. The data shows that students had no troubles using definitions to recognize objects but could not use definitions to construct them

The data suggests that there is a gap between the abstract entity students have constructed from the definition and the articulation process, the recognition per se. Another important issue that came from the analysis of students' responses is the use of quantifiers and understanding of quantification in general.

Quantifiers

The study did not intend to explore students' discourse or use of quantifiers. However, this problem could not be disregarded. Some students who participated in the study did not use quantifiers at all when defining objects. Sometimes, missing quantifiers did not mean that the concept was not recognized or used properly during problem solving process. The preliminary analysis of the interviews suggested looking more carefully at the written work in terms of the presence of quantifiers. Students used quantifiers more often when writing statements but sometimes students changed the order of quantifiers they used. For example, instead of writing $\exists\forall$ statement they had $\forall\exists$ statement (Figure 5). Quantification question is very important for concept formation and requires more exploration.

Conclusions and implications: The study showed that one needs to have previously abstracted ideas to understand a new abstract structure. Moreover, data analysis and further discussion revealed that an abstract concept cannot be learned without concrete examples and problems that involve the concept. In other words, the articulation of an abstract concept is required for coherent structure formation.

At the first stage of the learning process, students are often given a definition of a concept being studied. Sometimes several simple examples precede the definition. These activities give students a chance to generate a preliminary set of objects for assembling. All these objects are previously learned abstract ideas. The process of assembling is followed up by the process of theoretical generalization. Since a definition usually gives only a preliminary set of ideas for assembling, it is most likely impossible to coherently understand inner connections between the ideas and form a plausible abstract entity, which means that we deal with a preliminary generalization. The next standard instructional step is illustration of the concept via various examples. During this stage students are getting the first articulation experience and make first attempts to concept recognition. At this stage a student should be able to exemplify and counter exemplify the concept. It means that when the concept is learned, the process of abstraction of these objects gets into the following static form: 1) connected assembled ideas; 2) complete understanding of meaningful inner connections; 3) and open-minded recognition of the object. At this stage, a student also should be able to interchange from object recognition to assembled ideas, if needed. It follows that all stages of abstract concept formation are interconnected. There is a constant interaction between processes (assembling and articulation) within the process of abstraction. This observation implies that if there is a problem with one process the abstract concept cannot be appropriately formed. This discussion leads to the following summary of possible predicaments for concept formation: 1) Empirical generalization and abstraction instead of theoretical. Students are trying to learn concepts by extracting commonalities from given concrete objects and examples. 2) Assembling of unsuitable ideas. Students mistakenly assemble some ideas which are not supposed to be assembled to learn a certain concept. As a result, theoretical generalization results in a misleading abstract entity and further in false conclusions which look true under students' arguments. 3) Insufficient number of assembled ideas. 4) Making the object of recognition (during problem solving) one of the ideas for assembling. 5) Insufficient articulation. Students find it difficult to provide examples, especially counterexamples. 6) Isolation of concrete examples from objects of assembling. Sometimes students do not see the interaction between the concrete examples and the abstract structure. A concrete example is considered to be a static object with fixed properties.

Awareness of these predicaments can help to create meaningful instructional activities and classroom settings, giving enough examples and time so that students can articulate the concept they study. The theoretical conclusions can be applied to different mathematical courses at various levels. They are not limited by mathematics only and can be applied in other areas of study. To elaborate on these predicaments, more exploration, possibly within a different mathematical content, is needed.

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Figure 1: Process of Abstraction

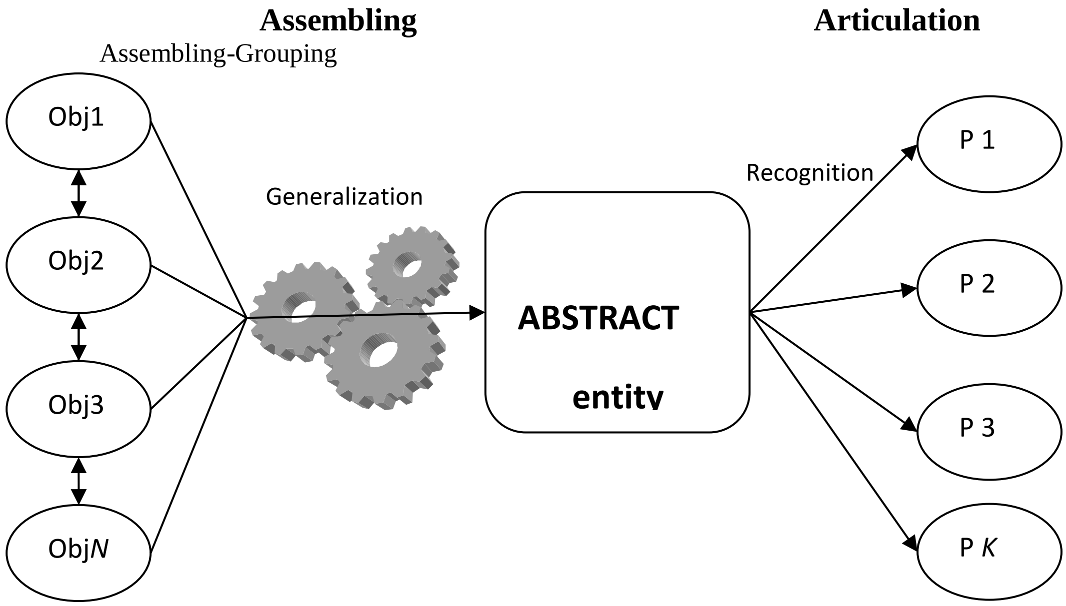


Figure 2: Problem with Closure

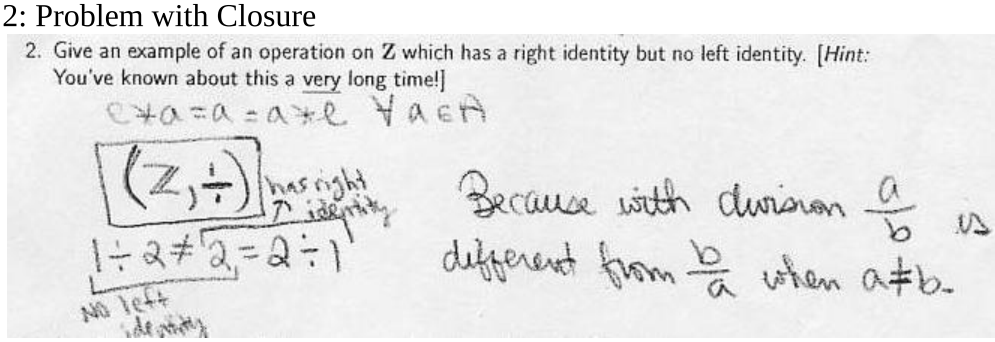


Figure 3:

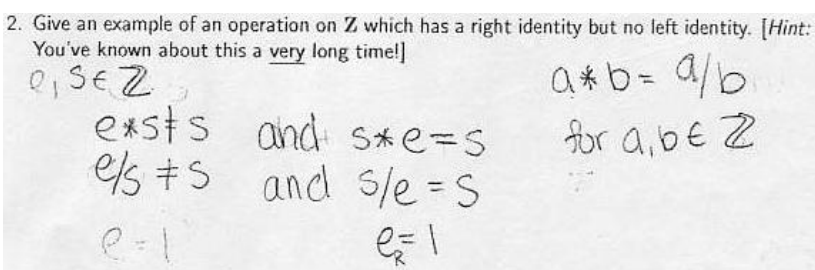


Figure 4: Student's response to the question: *Is it possible to find two nontrivial subgroups H and K of (**Z**, +) such that H∩K={0}? If so, give an example. If not, why not?*

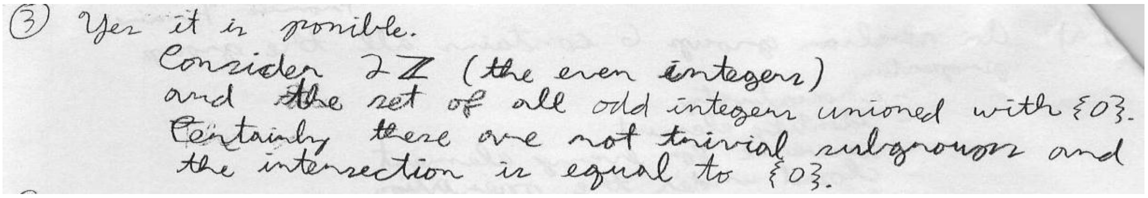
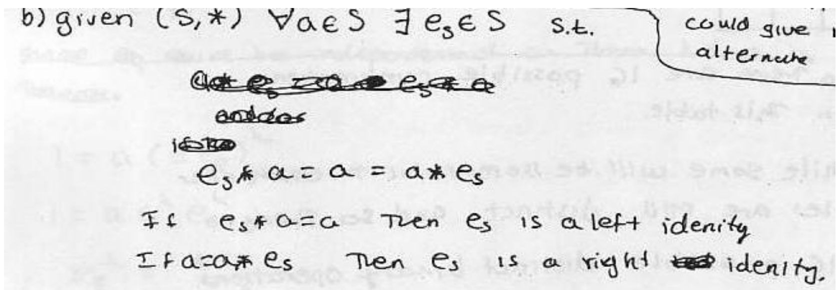

(3) Yes it is possible.
Consider 2Z (the even integers)
and the set of all odd integers unioned with {0}.
Certainly these are not trivial subgroups and
the intersection is equal to {0}.

Figure 5: Student defines an identity

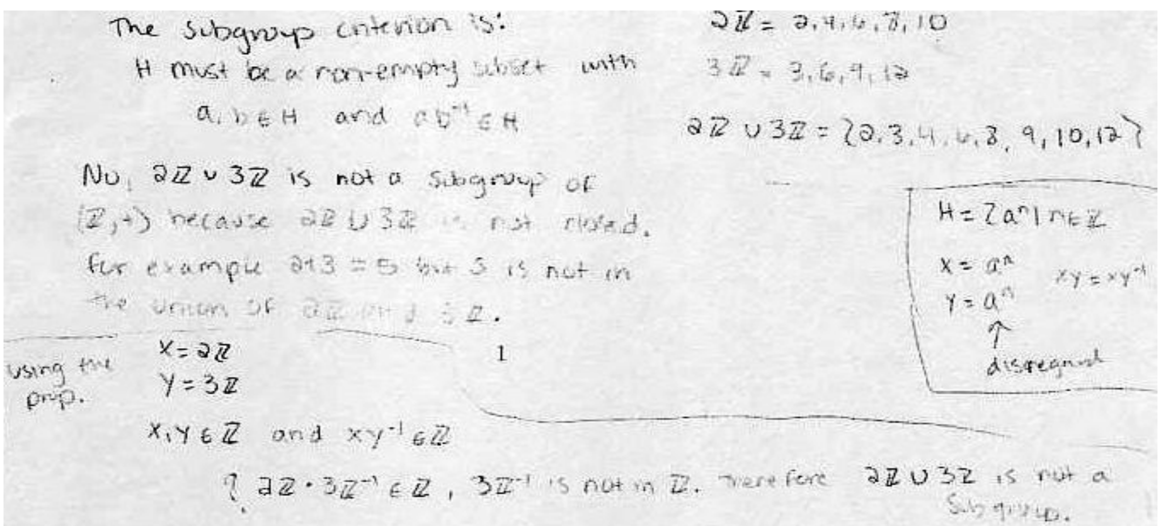

b) given (S,*) $\forall a \in S\ \exists e_s \in S$ s.t. could give alternate

$e_s * a = a = a * e_s$

If $e_s * a = a$ Then e_s is a left idenity

If $a = a * e_s$ Then e_s is a right idenity.

Figure 6: Switching from group operation (addition) to another operation (multiplication):


The subgroup criterion is:
H must be a non-empty subset with
a, b ∈ H and $ab^{-1} \in H$

$2\mathbb{Z}$ = 2, 4, 6, 8, 10
$3\mathbb{Z}$ = 3, 6, 9, 12
$2\mathbb{Z} \cup 3\mathbb{Z}$ = {2, 3, 4, 6, 8, 9, 10, 12}

No, $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subgroup of ($\mathbb{Z}$,+) because $2\mathbb{Z} \cup 3\mathbb{Z}$ is not closed.
for example 2+3 = 5 but 5 is not in the union of $2\mathbb{Z}$ and $3\mathbb{Z}$.

H = {a^n | n ∈ $\mathbb{Z}$}
x = a^n $xy = xy^{-1}$
y = a^n
↑
disregard

1

using the prop.
x = $2\mathbb{Z}$
y = $3\mathbb{Z}$
x, y ∈ $\mathbb{Z}$ and $xy^{-1} \in \mathbb{Z}$
? $2\mathbb{Z} \cdot 3\mathbb{Z}^{-1} \in \mathbb{Z}$, $3\mathbb{Z}^{-1}$ is not in $\mathbb{Z}$. Therefore $2\mathbb{Z} \cup 3\mathbb{Z}$ is not a subgroup.