SECONDARY TEACHERS' DEVELOPMENT OF QUANTITATIVE REASONING

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This study was designed to document the development of teachers' ways of thinking about quantitative reasoning, one of the standards for mathematical practice in the Common Core State Standards. Using a Models and Modeling Perspective, the authors designed a model-eliciting activity (MEA) that was implemented in a graduate mathematics education course focusing on quantitative reasoning. Teachers were asked to create a quantitative reasoning task for their students, which they subsequently revised three times in the course after receiving instructor, peer, and student feedback. The MEA documented the development of the teachers' models of quantitative reasoning, and this report details one group of three teachers' development over the course. Findings include an overall model of teachers' development that is both generalizable and sharable for other researchers and teacher educators.

Key words: In-Service Mathematics Teachers, Model Eliciting Activity, Quantitative Reasoning

Introduction and Literature Review

The Common Core State Standards (CCSS) are set to bring a new wave of reform measures for classrooms across the United States, and with this comes new goals for teachers and teacher education programs. Research about how teacher education programs can support these goals are lacking, especially concerning how programs align with the CCSS in ways that productively impact teacher practice (Confrey & Krupa, 2010; Sztajn, Marrongelle, & Smith, 2011). Literature addressing the gap between teacher practice and teacher education efforts has described how a models and modeling approach to in-service teacher education can challenge teachers to develop ways of thinking to help their students while simultaneously documenting the development for research purposes. This approach uses Model Eliciting Activities (MEAs), which are tasks that engage teachers in thinking about realistic and complex problems embedded in their practice in order to foster ways of thinking that can be used to communicate and make sense of these situations (Doerr & Lesh, 2003; Lesh & Zawojewski, 2007). MEAs have been shown to contribute to teacher development because these activities make teachers engage in applicable mathematics, consider student reasoning more deeply, and reflect on beliefs about problem solving (Chamberlin, Farmer, & Novak, 2008; Lesh, 2006; Schorr & Koellner-Clark, 2003; Schorr & Lesh, 2003).

While these studies have implemented successful MEAs for teachers, there is a need for additional activities given the recent demands the CCSS place on teacher education programs (Confrey & Krupa, 2010; Garfunkel, Reys, Fey, Robinson, & Mark, 2011). For instance, no MEAs currently exist that aim to identify and document the development of teachers' thinking about the CCSS standards for mathematical practice or the related area of quantitative reasoning. The purpose of this study is to investigate teachers' ways of thinking about quantitative reasoning by implementing a MEA in a graduate setting for secondary mathematics teachers. The specific research questions were (a) how do teachers' models of quantitative reasoning develop through a MEA grounded in their classroom practice? and (b) What researcher model can be developed from this process in order to produce generalizable and sharable findings for others?

Methods

The theoretical perspective we used for the study is a Models and Modeling Perspective, as described by Lesh and colleagues. In addition to having a powerful lens for examining teacher education, a Models and Modeling Perspective also provides guidelines for the methods that support significant findings given the current research questions. Given these methods, a Models and Modeling Perspective offers a framework for understanding teachers' ways of thinking, their development, and provides a mechanism for analyzing and piecing together findings (Koellner-Clark & Lesh, 2003; Hiebert & Grouws, 2007; Silver & Herbst, 2007; Sriraman & English, 2010).

The setting for this study was within a master's program in mathematics, where teachers took a combination of mathematics and mathematics education courses over two years; however, part of the study involved piloting the teachers' MEA in summer undergraduate mathematics courses. We focused the study on a newly developed mathematics education course in the program, called Quantitative Reasoning in Secondary Mathematics, which was offered Summer 2012. The authors designed the MEA, worth 50% of the course grade, to have the 21 teachers enrolled create and refine a quantitative reasoning task for their students with the intention of implementing the task in the fall. Teachers worked in groups of three or four and received feedback about their task during the summer from several sources, including the instructor, from each other, and from undergraduate students who completed the task. Each type of feedback prompted an updated iteration of the task and supporting documents that captured how the teachers' ways of thinking develop. Data collection consisted mostly of the iterations of documents generated by the MEA (see Table 1), with observations of Group 1 during in-class time devoted to the MEA. Using content analysis on the documents, the researchers identified patterns in the ways teachers' thinking about quantitative reasoning tasks developed due to this process. While the full study will analyze all 6 groups, here we present an analysis of one of the groups, which we call Group 1.

Table 1. Summary of Quantitative Reasoning (QR) documents analyzed

Assignment Name	**Short Description of Components**
Pre-Assignment	Document including initial models of QR, QR tasks, QR course
Version 1	Four documents including (a) Quantitative Reasoning Task; (b) Facilitator Instructions; (c) Assessment Guidelines; (d) Decision Log
Instructor's Feedback	Instructor's comments and suggestions to Version 1.
Version 2	Updated Version 1 in response to the instructor's feedback.
Teachers' Feedback	Groups swap Version 2 and offer comments and suggestions
Version 3	Updated Version 2 in response to the teachers' feedback.
Undergraduate Work	Student work after completing QR task (part (a) of Version 3)
Version 4	Updated Version 3 in response to student work, plus evaluation of student work.

Findings

Originally in their pre-assignment, the three teachers in Group 1 had wildly significant different definitions of quantitative reasoning, from drawing justified conclusions in real-world problems (Nicholas), visualizing amounts and number sense (Joyce), to using logical mathematical statements and deductive reasoning to reach conclusions using algebra (Percy). After reading the CCSS definition, all three teachers were asked to interpret what they thought this definition meant, for example by explaining what contextualization and decontextualization could look like in a secondary mathematics classroom. Nicholas chose to adopt this kind of language when asked to describe what quantitative reasoning looked like in his own classroom by saying "my students analyzed real-world problems and/or visual representations using contextualized mathematical representations." This statement reflects his earlier model of quantitative reasoning which he believed was evidenced by students' ability to determine the reasonableness of answers when solving real-world problems. The other two teachers expanded their original models without this type of language, identifying quantitative reasoning as focusing on relationships between quantities and making sense of what is being done when solving problems.

The group statement about quantitative reasoning given in Version 1 reflected the commonalities among the individual interpretations of the CCSS definition. Quantitative reasoning was defined as focusing on quantities when solving a problem, creating relationships between quantities, understanding why something works, and justifying the solution with units. These descriptions were all framed in terms of the logarithmic context they chose to develop for their MEA. All of these components were part of at least one teacher's definition of quantitative reasoning in the pre-assignment, with the exception of the incorporation of units, which Joyce was observed to adopt during the first week of class. The ideas about quantitative reasoning that did not persist into Version 1 were ones that only one of the three mentioned in the pre-assignment, such as Joyce's notion of number sense and Percy's incorporation of argument and deductive logic. The group connected their task to quantitative reasoning by having students demonstrate "what the quantities associated to a logarithmic function represent" and compare how these quantities compare in terms "big changes vs. small". The instructor feedback to these teachers largely encouraged the task's alignment to MEA principles, and the group was encouraged to continue documenting how they thought quantitative reasoning related to the task.

The teachers made a number of changes to the task in response to the instructor feedback, documenting the details and rational for the alterations in Version 2. The teachers claimed their Version 2 task "will encourage students to think quantitatively (about how quantities relate to each other in different scenarios – exponential vs. logarithmic) and also develop an understanding of logarithms" by having students model about interest rates by modeling the situation using both exponential and logarithmic functions. This model of quantitative reasoning was observed to develop during group discussions and may perhaps be a result of the instructor's in-class encouragement for the group to connect inverse functions and compare the size of various quantities.

The peer feedback provided information on how the teachers thought about quantitative reasoning through their evaluation of another group's task. Comments from the small group discussion that occurred in class were echoed in the peer evaluation document; overall, Group 1 thought Group 6's task was "a great opportunity to provide evidence of quantitative reasoning", particularly through the prompting questions included in the task. These questions included "What quantities would be represented in your explanation?" and "How are the quantities related to each other?" Group 1 concluded that after the evaluation process, they began "thinking differently about how we would like our students to work." Group 2 gave Group 1 feedback focusing on making the role of quantities more explicit in the task, and to make the students "look at all of the quantities more in depth…have them talk about each quantity and how it will relate to the situation and formula they're supposed to come up with." Aside from advice on how to improve the facilitator guidelines, Group 2 requested "a little bit more about your thought process regarding how you decided that this task could show quantitative reasoning in each of the students…maybe you could be more specific about this."

In response to the peer feedback, Group 1's Version 3 included changes addressing the comments, such as editing a question to ask students to "describe the co-varying relationship between the quantities that your identified variables represent". These and other changes indicate Group 1 incorporated suggestions from their peers while maintaining the ideas they previously had about quantitative reasoning. When addressing the comment asking how the task incorporated quantitative reasoning, Group 1 indicated a quantitative understanding of logarithms looked "like 'the answer to the logarithm is what exponent I would need to use on this base to make it into this number (the argument); or our visualization of the behavior and characteristics of logarithmic graphs; etc." This indicated the group model of quantitative reasoning now included a conceptual understanding of a mathematical topic, incorporating multiple representations of a contextualized nature. This change reflected the real-world context that had been part of the group's definition since Version 1, and the multiple representations that had always been a part of the task.

After piloting Group 1's MEA with three undergraduate students in a Concepts of Calculus course, the group's reaction to the work provided some of the richest data about how their individual models had developed. The in-class discussions typically began with a teacher bringing up a comment about an area where students had difficulties, errors, or gave unexpected responses. The group's conversation then tended to move towards identifying changes they would want to make in response to student difficulties. This evaluation process encouraged teachers to reveal the intent of each question and whether the teachers thought the student response addressed the goal of the question and the overall MEA to show student thinking about quantitative reasoning. The teachers made comments such as "[I'm] trying to put ourselves in the mind of the learners", indicating their transition to seeing the activity from a student perspective. These observational findings were triangulated through Version 4, particularly in their comments that "as we looked over our student feedback, it became apparent that our main goal was to improve questions that did not [elicit] the desired response from the students." These revelations from the teachers often resulted in changes being made to the questions to better meet the MEA and question goals.

Group 1 stated the changes in Version 4 were

…making the table go up by more consistent increments, streamlining language somewhat (rule vs. model vs. function vs. relationship vs. co-varying), and providing more guidance for the process of estimating an exponent solution. We also had discussion over whether this activity actually promotes and assesses quantitative reasoning, and we believe that ultimately it does. Students model their understanding of exponential functions with a table, graph, and function rule. They also do the same with the inverse, the logarithmic function. A significant motivator for creating this activity is students' anemic procedural understanding of logarithms.

The definition of quantitative reasoning and its connection to the task were similar to that stated in Version 3; however, their interpretation of student work included newer components. By evaluating students' quantitative reasoning, the teachers revealed the following attributes constituted evidence of the term: writing the relationships of quantities in words, explaining relationships between functions, algebraically working to contextualize and articulate quantities, and explaining mathematical observations through quantities. The teachers were also able to articulate what quantitative reasoning would look like in their own classrooms:

"Overall, the revision process was very valuable in creating the lesson. In particular, working with peers and discussing the student feedback was very worthwhile. The realistic and honest answers demonstrated how the students interpreted the investigation, and therefore created an opportunity to create a better lesson. We believe that the quantitative reasoning skills we are trying to develop are not only dependent on well conceived lessons, but almost more importantly, well conceived classroom attitudes and expectations."

Discussion

The development of this group's model of quantitative reasoning began with the consolidation of different definitions from each individual, where common characteristics between ideas were preserved and non-shared ideas were largely abandoned. These changes likely resulted from the classroom readings and activities describing frameworks for thinking about quantities and quantitative reasoning. For example, the instructor presented Moore, Carlson, and Oehrtman's (2009) definition for quantitative reasoning as attending to and identifying quantities, representing the relationships between quantities, and constructing new quantities during the first week of class; this definition is similar to the one submitted in Version 1. Similarly, the inclusion of Thompson's (1989; 1994; Smith III & Thompson, 2008) framework for quantities can be attributed to Group 1's inclusion of units in this definition. While characteristics of identifying and comparing quantities became clear in their definition, how these components were operationalized in the task were unclear.

The peer feedback process allowed the group to see other teachers' ideas of quantitative reasoning, and allowing the introduction and adoption of new ways of thinking. At the same time, the group was challenged to improve their task with comments from the peer feedback asking for further articulation of the task's connection to quantitative reasoning. Some of these changes were observed to be the result of instructor suggestions in class about how to operationalize these characteristics in terms of the group's MEA. By the third model, group expanded their definition to include core characteristics of the task they had included, such as conceptual understanding and multiple representations. The group more clearly stated the relation of their task to quantitative reasoning through these characteristics. This addition may be the result of the readings and assignments Group 1 completed from the Carlson and Oehrtman (2011) precalculus textbook, as prior to submitting this version Group 1 was exposed to ideas of multiple representations being used within a single problem about proportional reasoning and average speed, similar to the problems in Thompson (1994).

The student feedback was invaluable in promoting teacher development of models about quantitative reasoning, and about the task itself. These responses indicated the teachers' new ability to apply their definition of quantitative reasoning to a specific task for their students, in this case logarithms. The increased emphasis on connecting logarithmic and exponential functions may have resulted from the in-class activities that had teachers applying proportional reasoning to exponential functions, and contrasting these with linear functions.

These results have implications for both researchers and teacher educators. The use of this MEA pushed teachers to develop their model of quantitative reasoning. By the end of the course, teachers had moved from a range of definitions of quantitative reasoning to a more clearly defined model that connected this term to quantities, relationships of quantities, how these ideas were important to a context (financial planning using logarithms), and how these ideas could be developed in their students. Also, the role the instructor played by selecting reading and activities that introduced teachers to different frameworks about quantities, proportional reasoning, and exponential functions seemed to influence teachers' model development.

Given the influence of the CCSS on assessments taking place in the 2014 school year, teachers will be expected to include the standards for mathematical practice as a daily part of the mathematics classroom and connect these practices to content (CBMS, 2012; CCSS, 2010). The connection between quantitative reasoning and these practices mean teachers need to be able to interpret and instruct these ideas in meaningful ways. However, the findings indicate teachers have little experience with these terms, and interpret them in ways that are different and disconnected to classroom activities. From a practical stance, this MEA structure was successful for the goals of the course in that teachers were able to analyze the mathematical and conceptual structure of quantities and the relationships between quantities in secondary mathematics courses. For example, including seminal readings about this topic can contribute to encourage new ways of thinking about quantitative reasoning in teachers. Structuring teacher education and professional development to help teachers overcome these gaps in productive ways continues to be a major focus in mathematics education. This study may offer teacher educators and researchers a potential starting point for shaping teacher education in ways that support development of teachers' way of thinking about quantitative reasoning and other standards for mathematical practice.

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