COMMONLY IDENTIFIED STUDENTS' MISCONCEPTIONS ABOUT VECTORS AND VECTOR OPERATIONS

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Abstract

In this report we present the commonly identified error patterns and students' misconceptions about vectors, vector operations, orthogonality, and linear combinations. Twenty three freshmen students participated in this study. The participants were non-mathematics majors pursuing liberal arts degrees. The main research question was: What misconceptions about vector algebra were still prevalent after the students completed a freshmen-level linear algebra course? We used qualitative data in the form of artifacts and students' work samples to identify, classify, and describe students' mathematical errors. Seventy four percent of students in this study were unable to correctly solve a task involving vectors and vector operations. Two types of errors were commonly identified across the sample: a lack of students' understanding about vector operations and projections, and a lack of understanding (or distinction) between vectors and scalars. Final results and conclusions include research suggestions and practitioner-based implications for teaching linear algebra in high school and college.

Key words: Linear algebra, vectors, and students' misconceptions

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Background

Research suggests that students transitioning from computation-heavy courses to more proof-oriented mathematics courses have a lot of difficulties, especially in topics of linear algebra (Rasmussen et.al, 2010). Linear algebra serves as the bridge between many mathematics domains due to its content and significant connections between lower and upper level mathematics. Common Core Standards (CCSS) has placed a strong emphasis on students' learning of linear algebra topics such as vectors and matrices during their high school years to help students to better transition into business algebra, linear algebra, and calculus in college (CCSS, 2010).

However, the issue of conceptualizing abstract ways of reasoning is becoming increasingly problematic for most of the students (Rasmussen et. al. 2010; Hillel & Dreyfus 2005; Stewart & Thomas, 2009; Tabaghi 2010). Research studies have strongly suggested that students are able to grasp and perform the computational aspects; however, they have trouble understanding the conceptual notions and the mathematical ideas behind their computations (Stewart & Thomas, 2009, p. 951; Gueudet-Chartier, 2004; Tall, 2004). With linear algebra becoming a strong emphasis of high school mathematics curriculum (CCSS, 2010), consequently affecting many high school students and freshmen entering college, more research is needed to better understand students' difficulties and mathematical misconceptions in linear algebra.

Theoretical Perspectives

Whether it is due to a lack of visual representation or the task of generalizing familiar concepts, dealing with forms of abstraction and proof appears to be very difficult for students (Rasmussen et. al p. 1577, Hillel & Dreyfus p.181). Stewart and Thomas (2009) studied a group of undergraduate students, who struggled with vector addition and spanning, as well as not being able to distinguish between linear combinations and linear equations. The authors concluded that perhaps the teaching methods of linear algebra need to be re-focused to emphasize the "embodied world, symbolic world, and formal world" of mathematics (Stewart and Thomas, 2009, p. 956). These three worlds were initially introduced by David Tall (2008), who defined them as: the *conceptual-embodied* world - based on perception, action, and thought, the *proceptual-symbolic* world – based on calculation and algebraic manipulation, and the *axiomatic-formal* world – based on set-theoretic concept definitions and mathematical proof (Tall, 2008).

Tabaghi (2010) suggests that students' transition between operational thinking to structural thinking is critical. However, teaching this transition is difficult considering most mathematics instruction up to that point focuses on procedures and algorithms, leading many students to mainly develop *operational thinking*. Tabaghi defined operational thinking as "conceiving a mathematical entity as a product of a certain process" while *structural thinking* involves the conception of mathematical entity as an object (p. 1507). Tabaghi (2010) found that most of her students described linear transformation only as vectors (operational thinking). She explained that "typically, students are unable to visually represent the concept or do not adequately picture all the possibilities" (Tabaghi, 2010, p. 1507).

In contrast, Harel (1989) found that his students had difficulties in proof because they held a limited point of view about what constitutes justification and evidence. Harel (1997) claimed that students do not develop adequate concept images for the definitions provided by teachers or textbooks. Harel explained that, most textbooks use "algebraic embodiments rather than geometric ones", which are in part problematic due to their puzzling notion of "unfamiliar algebraic systems" (Harel, 1997, p. 56)

In this study we focused on students' error patterns and mathematical misconceptions related to vectors, vector operations and dilations, orthogonality, and linear combinations.

The main research question we sought to investigate was: *What misconceptions about vector algebra are still prevalent after the students complete a linear algebra course?* We collected students' written work samples from summative assessment at the end of the course to be able to classify, describe, and document students' error patterns and mathematical misconceptions related to these topics. Based on the findings of this research, we provided research suggestions and practitioner-based implications for high school teachers and educators, who are working to help students improve their experiences learning linear algebra.

Research Methodology

We used *phenomenography* as the research methodology for this study, which investigates the qualitatively different ways in which people think about something (Marton, 1981; 1986). Qualitative data collection methods were used to capture students' thinking and understanding of concepts of vector algebra. We collected hand-written student work samples and artifacts focusing on constructive response questions that required students to provide answers as well as explanations to their answers. The goal was to collect data in such a way that the researchers imposed a minimal amount of mathematical influence or instructional bias on the participants and their data. The researchers were not the instructors of this course nor were they familiar with the students enrolled in the course.

Methods & Procedures

To ensure better-quality instruction, careful consideration was given to the selection of the linear algebra course and its instructor. The instructor of the selected course was a senior professor of mathematics with expertise in linear algebra, who had taught the course for many years and was the author of the course textbook.

Sample & Context

The participants of this study were twenty three first-year college students enrolled in a Linear Methods course in a small urban private liberal arts college. Linear Methods is a variation of a traditional Linear Algebra course. This course was developed in response to a new curriculum for incoming college freshmen. The student participants were liberal arts majors (e.g., business, social sciences); mathematics and science majors do not take this course. As a requirement, students must take one mathematics "beauty" course, that is not computation-heavy (such as calculus), introducing students to a more elegant aspect of mathematics. All the "beauty" courses are required to include explorations of basic concepts of logic and methods of proof. These types of courses are considered tamer versions of their advanced counterparts, such as: Number Theory, Differential Geometry, Topology, and Abstract Algebra. As part of the course, our participants touched upon some essential topics of linear algebra without delving far into the theory. The class met for a total of fifty minutes three times per week for eleven weeks. It was taught by a senior professor of mathematics in a lecture format with very little small group student interactions. The instructor encouraged students to attend supplementary review sessions organized and taught by the teaching assistant two times per week.

Linear Methods Course

In Linear Methods students studied several linear algebra topics, such as: vector manipulations, scalar multiplication and vector addition, systems of linear equations, inverse matrices, and linear transformations (e.g., rotations and reflections over lines). The instructor taught these topics mostly from a deductive perspective by concentrating students' attention and learning on the definitions and formal structures. For example, the curriculum is organized to sequence matrix addition before introducing vectors, and later in the course defines a vector as a special type of matrix. The curriculum includes very few opportunities for students to explore the reasons for why two matrices (and ultimately vectors) can be added or multiplied together, while focusing largely on the mathematical procedures and examples for students to practice mathematical computations of matrix (and vector) manipulations. Over the past few years, however, the instructor had expressed concerns that the students in this course are fundamentally struggling with graphing exercises, especially the principles of vector addition and scalar multiplication. He also indicated that the students lack understanding and do not see the connections between matrices and linear systems.

Mathematical Task

The following task was the main question of our students' data and analyses:

Express the vector $\vec{x} = [x_1, x_2, x_3]$ ***as a sum of two vectors, one of which is parallel to*** $\vec{y} = [y_1, y_2, y_3]$***, and second is orthogonal to*** $\vec{y}$ ***. Use fractions instead of decimals.***

This task required students to present their work and demonstrate an understanding of six overarching mathematical concepts related to vectors:

- Finding a parallel vector to a given vector (i.e. shift of a vector)
- Finding a perpendicular vector to a given vector (i.e. dilation of a vector, the projection of a vector onto another vector)
- Vector multiplication by a scalar;
- Finding the length of a vector;
- Vector addition and subtraction;
- Representing vectors as a sum of two vectors;

We selected students' work from this task because it included multiple parts and necessitated students to illustrate their answers, which allowed us to thoroughly analyze their work and identify mathematical error patterns. This task was given to the students at the end of the course, as part of their final exam. Thus, students had a period of one semester to confront their misunderstandings of linear algebra by asking either the instructor or the teaching assistant assigned to the course.

Data Analyses

Our goal was to investigate student's responses, specifically their errors and misconceptions related to vector algebra. We coded students work for error patterns within the six abovementioned categories. We then analyzed the coded data for common themes of error patterns and misconceptions. We focused on the mathematical errors behind the solutions, rather than students' processes of obtaining the solution. We generally noted the arithmetic and computational errors in our analyses as well (i.e. a student multiplied two fractions by finding a common denominator), however we didn't focus our analyses on these types of errors. We also did not analyze (nor did we collect) the course grades of these students.

Findings

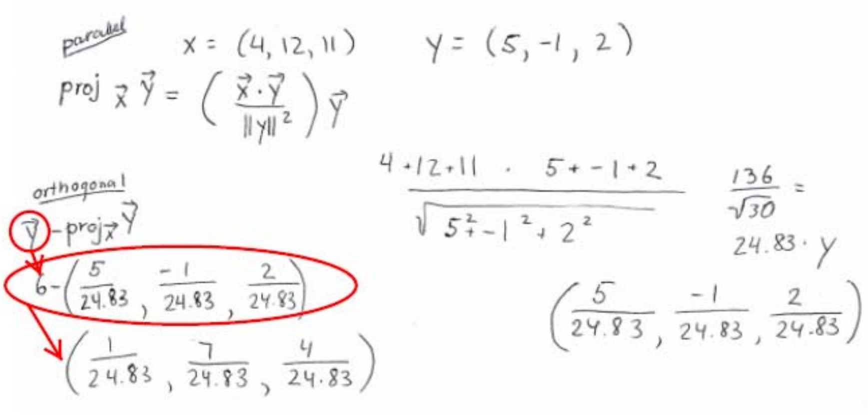
Only five students (out of 23 total) were able to correctly answer the question. One student did not provide a response to the question. The remaining seventeen students answered the question exhibiting two types of common misconceptions related to: the reasoning and spatial sense about vector operations and projections, and the understanding (or distinction) between vectors and scalars.

Lack of Understanding (or Distinction) Between Vectors and Scalars

Thirty five percent of students (8 out of 23) demonstrated fundamental misunderstandings of the meaning of vectors and scalars, and failed to differentiate between vectors and scalars. These errors were especially evident in students' operations with vectors - students confused vectors with scalars and performed arithmetic operations, often treating them as numbers. Many students in this category also have mistaken the vectors for scalars and used algebraic operations with them to obtain either vectors or scalars as a result.

For example, in Figure 1, Bobby computed a difference between a scalar and a vector and got a vector as a result.

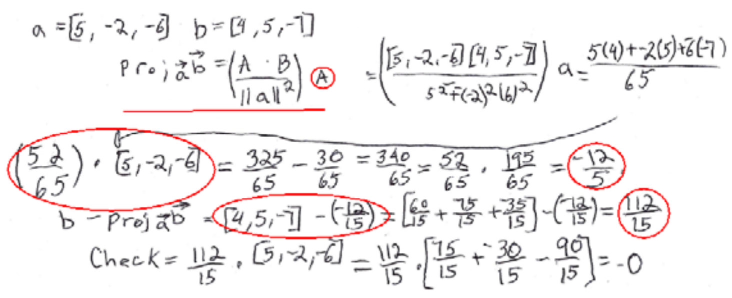
***Figure 1*. Bobby's Misconception: Scalar – Vector = Vector**



Other errors were also evident in Bobby's work, including: $\vec{y} = 6$; assumption that vector = scalar; incorrect reasoning about the projection of vectors; subtraction of fractions from whole numbers; calculation errors (136 in the numerator); and the use of square roots.

In contrast, Figure 2 illustrates the work of Casey, who subtracted a scalar from a vector and got a scalar as the result.

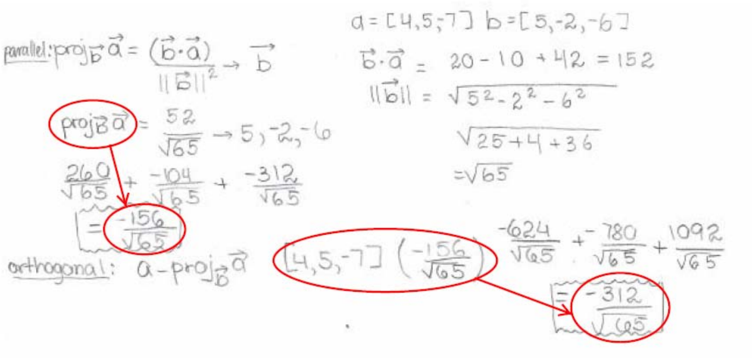
***Figure 2*. Casey's Misconception: Vector – Scalar = Scalar**



Similarly, Casey also used incorrect reasoning about: projection of vectors; assumption that vector = scalar; and a false interpretation for the dot product of a vector and a scalar.

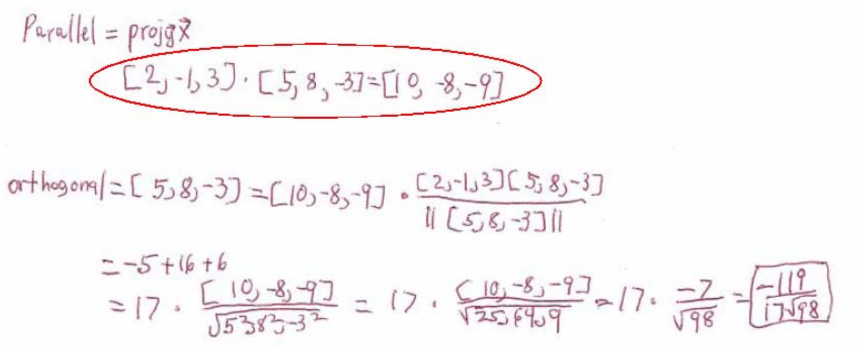
Indeed, the dot products of vectors and scalars have been common error patterns for most of our students' work samples. For example, Harper interpreted the dot product of a scalar and a vector as a vector. Harper, another student, also incorrectly assumed that vector = scalar, and used the square roots in the solution of this problem (see Figure 3).

***Figure 3*. Harper’s Misconception: Scalar • Vector = Scalar**



The last work sample (Figure 4) in this category that we chose was the work of Hayden. This student demonstrated many difficulties and misconceptions.

***Figure 4*. Hayden’s Misconception: Vector • Vector = Vector**

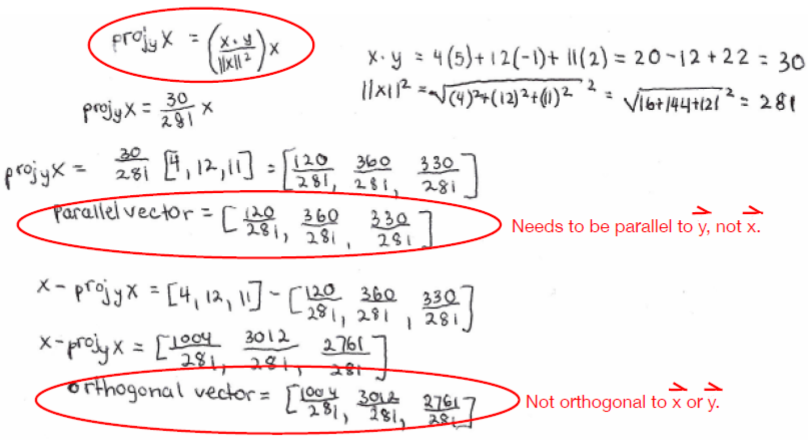


It was evident that Hayden struggled with simple arithmetic computations (multiplication of fractions by whole numbers), in addition to the false assumption that vector = scalar, incorrect reasoning about parallel and perpendicular vectors, and calculating the result of a dot product of two vectors as a vector (Figure 4).

Lack of Reasoning and Spatial Sense about Vector Orthogonality & Projections

Forty percent (9 out of 23) of students demonstrated limited understanding about the concept of parallel and perpendicular vectors. The common error was the fact that the students interpreted the vector projection as a scalar. These misconceptions were evident from the students’ work samples above; however, we present one more example to strongly emphasize these misconceptions (see Figure 5,where $\vec{x}$ [4, 12, 11] and $\vec{y}$ [5, -1, 2])

***Figure 5*. Jordan’s Misconception: Parallel and Perpendicular Vectors**



The results of these findings, also suggest that perhaps students used incorrect interpretation for the vector projections due to the lack of their understanding of the notation for vector projections. Another hypothesis is that students may have possibly confused the spatial orientation of the vector that they needed to project. Nonetheless, these errors demonstrate students' fundamental misconceptions about vectors, orthogonality, and the meaning of vector projections.

Discussion & Conclusion

We wanted to stress the fact that the students in our sample were freshmen college students completing a "beauty" mathematics course in linear algebra designed to meet the general education program course requirement for non-mathematics majors. One of the aspects that stood out in the students' work across the entire sample, however, was the fact that none of them used pictures to represent (or reason through) the solution to this task. As part of the recommendations for high school and entry-level college teaching of linear algebra, we would like to propose additional approaches that might be helpful for students' learning, especially approaches emphasizing pictorial representation.

Recent research strongly emphasizes the use of geometric approaches and representations in linear algebra. Gueudet-Chartier (2002) investigated students' geometric intuition, "use of geometrical or figural models", and its effect on students' ability to find mathematical models and develop conceptual understanding. The author suggested that mathematics instruction that focused on the use of drawings in general vector spaces was critical for his students; otherwise, the students were unable to find models and correct intuition to develop conceptual understanding of these topics (Gueudet-Chartier 2002; 2004).

Similarly, Harel (1989) found that, in comparison to strictly algebra-taught students, the students, who engaged in geometric interpretations outside of just algebraic ones, were able to answer more questions correctly and had an easier time visualizing and understanding the concepts using concept images.

Geometric representations also help to develop structural thinking (Tabaghi, 2010). Tabaghi argued that, typically, students are unable to visually represent the concept or do not adequately picture all the possibilities (Tabaghi, 2010, p. 1507). Therefore, incorporating opportunities for the students to explore abstract concepts through visual representations is critical and necessary to help students overcome the difficulties and misconceptions in linear algebra (Tabaghi, 2010).

One of the possible solutions of utilizing pictorial approach is included as a sample solution in Appendix A. This solution takes on an analytical (geometric) rather than procedural (computation-based) approach. First, this solution provides a geometric meaning that the projection of $\vec{x}$ onto $\vec{y}$ is parallel to $\vec{y}$ by shrinking/expanding $\vec{y}$ (thus the result is $c\vec{y}$, where c is a non-zero constant). Second, this solution emphasizes a geometric meaning that $\vec{x}$ minus the projection of $\vec{x}$ onto $\vec{y}$ is orthogonal/perpendicular to $\vec{y}$ (thus the dot product is zero). To help the students to “see” the vector (in blue), basic reasoning about geometric addition of vectors is needed. Additional prerequisite knowledge required for this task is: the distributive property of dot products and factoring out a constant.

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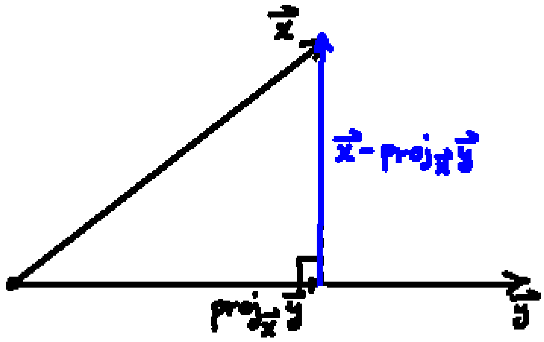
APPENDIX A



$$\underbrace{\text{proj}_{\vec{x}}\vec{y}}_{\text{PARALLEL TO } \vec{y}} + \underbrace{\left(\vec{x} - \text{proj}_{\vec{x}}\vec{y}\right)}_{\text{ORTHOGONAL TO } \vec{y}} = \vec{x}$$

SINCE $\vec{x} - \text{proj}_{\vec{x}}\vec{y}$ IS ORTHOGONAL TO $\vec{y}$, $(\vec{x} - \text{proj}_{\vec{x}}\vec{y}) \cdot \vec{y} = 0$.
ALSO, SINCE $\text{proj}_{\vec{x}}\vec{y}$ IS PARALLEL TO $\vec{y}$, $\text{proj}_{\vec{x}}\vec{y} = c\vec{y}$. NOW,

$$(\vec{x} - c\vec{y}) \cdot \vec{y} = 0$$
$$\vec{x} \cdot \vec{y} - c\vec{y} \cdot \vec{y} = 0$$
$$\vec{x} \cdot \vec{y} = c\vec{y} \cdot \vec{y}$$
$$\frac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}} = c$$

SO, $\text{proj}_{\vec{x}}\vec{y} = \dfrac{\vec{x} \cdot \vec{y}}{\vec{y} \cdot \vec{y}}\vec{y} = \dfrac{(4,5,-7) \cdot (5,-2,-6)}{(5,-2,-6) \cdot (5,-2,-6)}(5,-2,-6) = \dfrac{52}{65}(5,-2,-6)$.

FINALLY, $\text{proj}_{\vec{x}}\vec{y} = (4, \frac{-8}{5}, \frac{-24}{5})$ AND $\vec{x} - \text{proj}_{\vec{x}}\vec{y} = (4,5,-7) - (4, \frac{-8}{5}, \frac{-24}{5})$
$= (0, \frac{33}{5}, \frac{-11}{5})$.