FOSTERING STUDENTS' UNDERSTANDING OF THE CONNECTION BETWEEN FUNCTION AND DERIVATIVE: A DYNAMIC GEOMETRY APPROACH

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Students' difficulties with relating the graphs of functions to the graphs of their derivatives have been well documented in the literature. Here I present a Geometer's Sketchpad based applet, which was used as part of a technologically enriched Calculus I course. Individual interviews with students conducted after this in-class activity show evidence of varied and powerful student problem solving strategies that emerged after participation in the activity.

Key words: [Graphing, Technology, Derivative, Calculus, Geometer's Sketchpad]

Students' difficulties with graphing derivatives of functions have been a recurring theme in the calculus education literature. Given the recurrence of this theme it is unfortunate that only a few instructional interventions aimed at bridging this gap have been studied. Here I present a Geometer's Sketchpad (GSP) based applet that is designed to address a reoccurring reason for students' difficulty with graphing derivatives. Individual interviews conducted with students after this in-class activity revealed a diverse range of student-invented problem solving approaches for tackling novel graphing tasks.

Background Literature

A growing body of literature supports the assertion that a coordination of both visual-graphical and analytic reasoning is essential for students to form a rich understanding of mathematics (Aspinwall & Shaw, 2002; Zazkis, Dubinsky, & Dautermann, 1996). Zimmerman (1991) went so far as to state that, "visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject" (p. 136). Students' difficulties with graphing derivatives have been studied in the education research literature for at least thirty years (Orton, 1983; Nemirovsky & Rubin, 1992; Aspinwall, Shaw and Presmeg, 1997; Haciomeroglu, Aspinwall, & Presmeg, 2010). Some researchers have attributed these difficulties to pre-tertiary mathematics education that commonly deemphasizes the importance of graphing and graphical intuition in favor of more symbolic and algorithmic approaches to mathematics (Vinner, 1989). However, this places the blame elsewhere and makes students' difficulties with graphing a foregone conclusion, rather than an issue that can be addressed. Another series of explanations revolves around the coordination of different types of quantities. In order to take the Cartesian graph of a function (for which the function is not given) and sketch the graph of its derivative one needs to coordinate two very different types of quantities—the function's instantaneous rate of change, which is a gradient measure, and the height of the derivative function relative to the x-axis, which is a linear measure. Coordination of varying quantities is in general difficult for students (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Coordination of the gradient and linear quantities required for graphing derivatives can be viewed as a special case of this difficulty.

This outlook provides an explanation for the most common error that occurs when sketching the graph of a derivative, which is simply to redraw the original graph (Kung & Speer, in press; Nemirovsky & Rubin, 1992). Studies that have examined students working on these graphing tasks in clinical interview environments have revealed that students attended to the desired attribute of the original function, its slope, but coordinated it with the slope of the derivative function, leading to a redrawing of the original function, rather than the drawing of the desired graph. In essence, students coordinated a quantity with itself rather than coordinating two different quantities. A visual link that coordinates the correct two quantities is built into the design of the applet discussed in the following section.

The Applet and Associated Activity

The applet described in this section aims to provide a manipulatable link between gradients and linear measures, one that can be used both in association with functions and independently of functions. This is done through the use of slope-widgets. Each widget consists of two vertically aligned manipulatable points one of which has a line through it. The slope of this line is controlled by the height of the other point (see Figure 1 and Figure 2). When the slope point is placed on a function, the second point is manipulated so that the line approximates a tangent line. The widgets provide a tangible visual link between the slope of the function and the value of its derivative at that point. When several slope widgets are used in unison they plot out the path of the derivative function using this tangible link.

Students were given access to the slope widget applet during class time and asked to work in small groups to graph the derivatives of several functions with the aid of this applet.

Participants

One group of three students (two male and one female) was video recorded during group work throughout the semester. The group was chosen based on their scores on the Calculus Concept Readiness test (Carlson, Madison & West, Submitted) to be fairly representative of the class as a whole. Each member of the group also participated in a series of three individual problem solving interviews, conducted by the author of this paper, which included graphing tasks as well as other calculus tasks. These interviews were conducted in a technology free environment in which students had access to only pencil and paper. The data in the following section comes from the first set of interviews.

Data

The participants demonstrated a diverse range of strategies for approaching graphing tasks, particularly considering they worked with each other four days a week in class. One particularly interesting task, which illustrates some of this diversity, was given during the first set of interviews. Each of the students was given the graph of a function without being given its formula and told that the given function was the derivative of another function (Figure 3). They were asked to sketch the graph of the original function. This task was given early in the semester, when students had not yet encountered the concept of anti-derivative (or integral). The only instruction they had received which targeted graphing derivatives was the slope widget task described in the previous section. When the task was presented the students had only seen ‘graph the derivative’-type tasks, but not its ‘inverse’. I was interested in studying the problem solving strategies that students implement in approaching this novel-for-them task.

All three students, which will be referred to as Allison, Brad and Carson, where able to successfully complete the task fairly quickly (under three minutes) and then spent several minutes convincing themselves of the legitimacy of their solution. Each demonstrated a unique problem solving strategy.

Brad was the only student whose reasoning centred around the gradient-linear measure relationship.

> Brad [00:33:58]: [reading] Below is the graph of the derivative of a function sketch the graph of the original function. [begins sketching function] So from here on the original the slope is increasing until it crosses zero and then it begins decreasing and then it hits zero and after it hits zero it goes back up until it hits zero again.

Alison, in contrast, used a hybrid strategy starting with algebraic reasoning and adjusting her algebraic images based on graphical reasoning. She started with familiar graphical shapes by drawing a $y=x^2$ graph and a $y=x^3$ graph stating that one was the derivative of the other. Then she noted that unlike the $y=x^2$ shaped graph the given graph had a portion that was under the x-axis and that this portion of the graph corresponded to a portion of the original graph which had a negative slope. She then used this to adjust the shape of the x^3 graph appropriately.

Carson, demonstrated a different type of hybrid strategy. He first began by reasoning that the zeros of the given function corresponded to maxima, minima or saddle points. However, he had not yet encountered the appropriate terminology for these types of points and instead drew small sections of functions to describe each of these phenomena. He then expanded his reasoning to try to figure out which of the three situations he was dealing with. He started by reasoning that the derivative of a downward facing parabola is a line with a negative slope which crosses the x-axis at the same x-value as the vertex of the parabola. This was used to deal with the first zero of the given function. Then he reasoned that the derivative of an upward facing parabola is a positively sloped line that crosses the x-axis at the same x-value as the vertex of the parabola. Connecting these two shapes provided the desired cubic function shape.

Discussion

What is particularly interesting is that on the anti-derivative task two of the three students used strategies that incorporated both analytic reasoning and graphical reasoning. If I expand to look at other questions asked during the interviews, there is evidence of such thinking from all three of the studies participants. This is surprising given the limited experience these students had before the course with graphical modes of thinking. Students' difficulties with both subscribing to graphical modes of reasoning (Vinner, 1989) and translating between these and analytic modes (Apsinwall & Shaw, 2002; Haciomeroglu et al., 2010) has been documented in the literature. Such preferences for one mode of thinking over another are common enough that some authors draw fairly rigid distinctions between students who prefer either graphical or analytic modes of reasoning (e.g. Apsinwall et al., 1997; Haciomeroglu et al., 2010). The students in this study, partially due to their exposure to applets such as the one described above, moved between these representational forms relatively fluidly. In other parts of the interview data students were able to spot their own mistakes when one mode's results seemed to contradict another's. It is important to mention that these were not especially talented mathematics students. They were chosen to be fairly representative of the class as a whole. In fact Brad, in spite of putting considerable effort into his schooling, was unable to finish the course with a passing grade. The flexible thinking that these students demonstrated can be attributed to the types of powerful reasoning that the applet was able to foster. Further inquiry into the specifics of how students interacted with the applet in class will hopefully help me better develop a theory regarding how the applet affected these students' thinking about graphing of derivatives.

Questions

1) What theoretical framework can be used to analyze these data, and the rest of the interview data from my study? Will coordinating several frameworks be preferable?

2) Have you ever experienced phenomena with your students that are not in accord with the research literature? Was technology involved? How did you make sense of it?

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Appendix

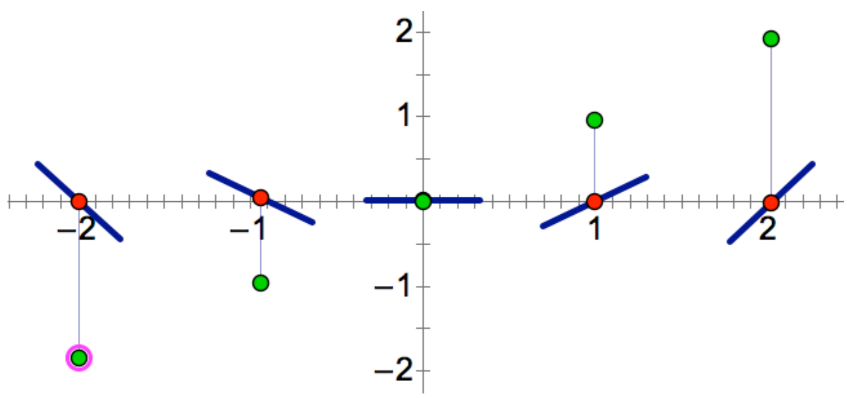

Figure 1: Slope-widgets

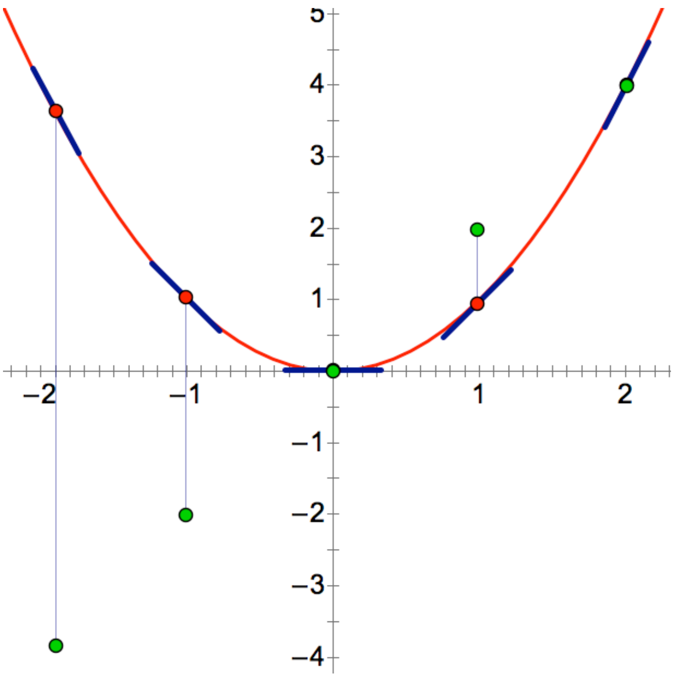

Figure 2: Slope-widgets on a function

Below is the graph of the derivative of a function. Sketch the graph of the original function.

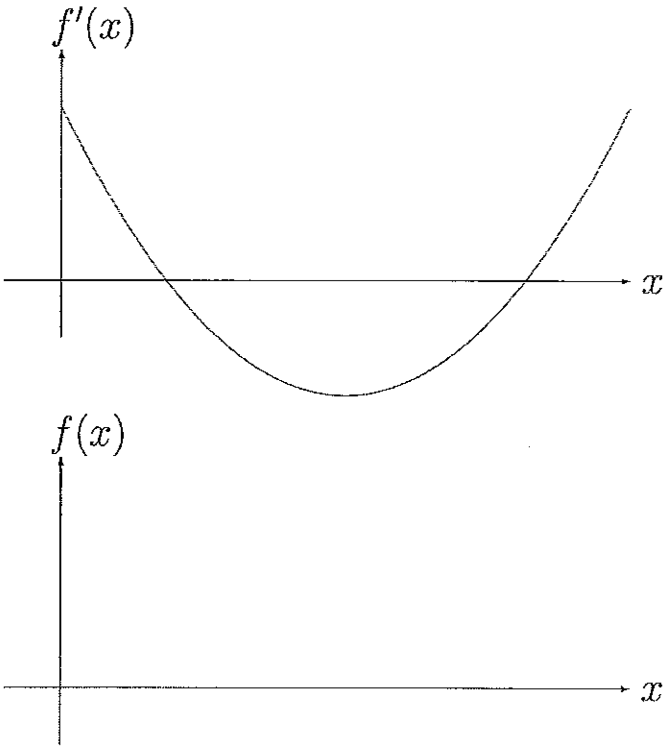

Figure 3: The anti-derivative task