DEVELOPMENT OF STUDENTS' WAYS OF THINKING IN VECTOR CALCULUS

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In this talk, we describe the development of the ways of thinking of 25 vector calculus students over the course of one term. In particular, we characterize the generalizations that students made within and across interviews. We focus on the construction of the semi-structured pre and post interviews, trace the construction of explanatory constructs about student thinking that emerged from those interviews, and describe how those constructs fit within the broader literature on student thinking in advanced calculus. We conclude by exploring implications for future research and practical applications for educators.

Key words: *Calculus, quantitative reasoning, rate of change, covariational reasoning, generalization.*

Background Literature and Research Question

The transitions that students make as they progress through the calculus sequence is important for both students' immediate success and their meaningful application of those concepts to other fields. The most important transitions that students make are in their understanding of function and rate of change that allow them to represent, predict and explain relationships between quantities in a system and in the generalization of those ways of thinking to systems with many quantities. However, research that explores those transitions is limited (Martinez-Planell & Trigueros, 2012; Trigueros & Martinez-Planell, 2010; Yerushalmy, 1997). Researchers have suggested that quantitative and covariational reasoning are foundational to students' coherent understanding of functions and rate of change (Carlson, Oehrtman, & Thompson, 2008; Saldanha & Thompson, 1998; Thompson, 1994; Weber, 2012). Studies have also shown the importance of quantitative reasoning to generalization (Ellis, 2007a, 2007b). This research led us to hypothesize that to study the development of students' ways of thinking about function and rate of change across the calculus sequence we would need to a) Focus on the transition between single and multivariable calculus, b) Characterize the development of the students' quantitative and covariational reasoning, and c) Understand the generalizations students make as they progress through the transition from single to multivariable calculus.

Given our hypothesis, we constructed a series of semi-structured interviews of students as they participated in vector calculus at a mid-size Northwestern institution. We used these interviews to gain insight into the major research question:

> *How do vector calculus students' ways of thinking about function and rate of change develop as they participate in a typical vector calculus course?*

This major research question led naturally to gaining insight into the role of quantitative and covariational reasoning as well as generalization in the development of those ways of thinking.

Theoretical Framework

Quantitative reasoning and generalization were the foundation for the construction of the semi-structured interviews and interpretation of the data gathered from them. Quantitative reasoning refers to a way of thinking that emphasizes a student's cognitive development of conceptual objects with which they think about mathematical situations (Smith III & Thompson, 2008; Thompson, 1989). Quantitative reasoning focuses on the mental actions of a student conceiving of that situation, constructing quantities in that situation, and relating, manipulating, and using those quantities to make a problem viable. Thompson (2011) argued that for a student to imagine that a function of two variables is a representation of the invariant relationship among three quantities, the student must construct those quantities, whether it is from an applied context or an abstract mathematical expression. As an example, if a student is to think about a complicated situation with three quantities and construct quantities and the invariant relationship between those quantities, a dynamic mental image of how those quantities are related is critical. That image positions a student to think about how quantity 1 varies with quantity 2, how quantity 2 varies with quantity 3, and how quantity 1 varies with quantity 3. Understanding these individual quantitative relationships allows a student to construct a function that expresses an invariant relationship of quantity three as a function of quantities one and two, so that its value is determined by the values of the other two. This example is a powerful characterization of how students might reason about situations with functions of many variables, and Ellis' (2007) generalizing actions framework provides a framework for how this extension might occur for a student. Her characterization of three generalizing actions: relating (Type 1), searching (Type 2), and extending (Type 3), served as a tool with which to construct the semi-structured interviews with a focus on quantitative reasoning, and to explore and categorize the generalizing actions students exhibited in those interviews. Together, Thompson's characterization of quantitative reasoning and Ellis' generalizing framework provided a foundation for exploring how students made generalizations over the course of their vector calculus course.

Method

We selected 25 students enrolled in vector calculus at a mid-size, Northwestern University. We chose the vector calculus course at this university because the course was the students' first exposure to multivariable functions in mathematics, which allowed us to observe the students' initial fits and starts with systems with many quantities. The students were randomly selected from all vector calculus students enrolled during that term, and were asked to participate in the study. They were compensated for their participation in the interviews.

Each student participated in a pre and post interview, and completed a number of problems during the vector calculus course. The pre and post interviews consisted of questions designed to gain insight into the students' ways of thinking about function and rate of change. The pre-interview focused on single-variable functions and rates, as the students had not yet been exposed to these ideas in their course. The pre-interview questions were identical across students. The problems they completed during the term were based on the progression of the vector calculus course and were identical for each student. These problems were open ended and designed to gain insight into the generalizations that students were making. The post-interview took place at the end of the term, and consisted of four common items, and four items based on the students' responses on the pre-interview and problems completed during the term.

Analysis of the data was multi-phased. We used the pre-interviews to characterize students' quantitative and covariational reasoning in explaining their ways of thinking about function and rate of change. We used open and axial coding to identify common behaviors and responses across interviews in constructing a number of ways of thinking that we hypothesized students had. As the students progressed through the course, we used the initial ways of thinking we identified in the pre-interview to understand how those ways of thinking played a role in the generalizations students made as they learned about multivariable functions and derivatives. We documented the development of the students' generalizations over the course of the term, and gained further insight into these generalization actions with the post-interviews. As a result, we were able to create a model for each student's way of thinking about function and rate of change and compare those models across students. This process of constructing inferences about student thinking allowed us to explain the role of quantitative reasoning in the generalizations students made as they transitioned from single to multivariable calculus.

Results & Discussion

Analysis of the data is ongoing, but our preliminary interpretations suggest that many of the students in vector calculus likely possess "flawed" understandings of functions and rate of change. By flawed, we mean that the students make inconsistent, and often incorrect statements about rate of change and interpretations of functions' graphs. We believe the students struggles are explained by their lack of ability to reason quantitatively. These findings are not necessarily surprising given the prevalence of literature that has documented students' difficulties with functions and rates, but we believe that their struggles are related to the type of generalizing actions that they exhibited during the course of the study.

We believe that the students' difficulties can be partially explained by their attention to calculational reasoning. By calculational reasoning, we mean that the students are concerned primarily with arriving at an answer without attention to the meaning of that answer in a particular context. The calculations reasoning that students exhibited constrained the type of generalizing actions they were able to perform. For example, their Type I and Type II (see Figure 1) almost entirely focused on the numbers and answers in a given problem. Rarely was the meaning of a particular mathematical idea the driving force behind a student's generalizing action. In the presentation, we will provide an overview of the students' actions in the context of calculational reasoning. We believe these insights support the categories defined by Ellis (2007), but also present an opportunity to expand on the generalizing actions framework.

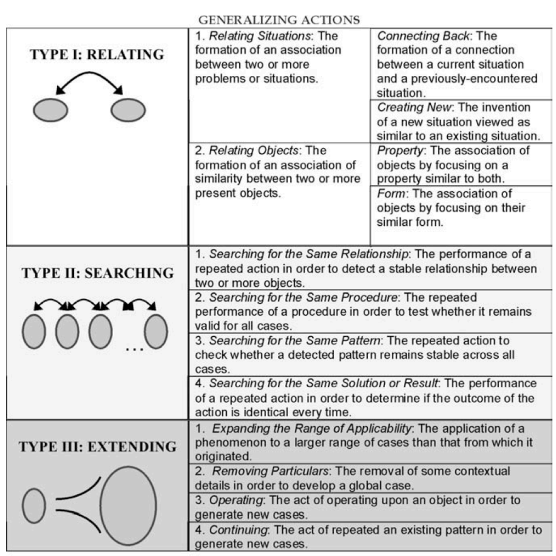

Figure 1. Ellis' (2007) framework for generalizing actions.

Questions for Audience

1) Based on the study's research question, in what ways could the research design be improved for future iterations of use with vector calculus students?

2) Based on the data presented, what were the strengths and weaknesses of the constructs we identified?

3) What are some essential questions you think must be answered to effectively characterize students' transitions from single to multivariable calculus?

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