COVARIATIONAL REASONING AND GRAPHING IN POLAR COORDINATES

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An extensive body of research exists on students' function concept in the context of graphing in the Cartesian coordinate system (CCS). In contrast, research on student thinking in the context of the polar coordinate system (PCS) is sparse. In this report, we discuss the findings of a teaching experiment that sought to characterize two undergraduate students' thinking when graphing in the PCS. As the study progressed, the students' capacity to engage in covariational reasoning emerged as critical for their ability to graph relationships in the PCS. Additionally, such reasoning enabled the students to understand graphs in the CCS and PCS as representative of the same relationship despite differences in appearance. Collectively, our findings illustrate the importance of covariational reasoning for conceiving graphs as relationships between quantities' values and that graphing in the PCS might create one opportunity to promote such reasoning when combined with graphing in the CCS.

Key words: Polar coordinates, Covariational reasoning, Graphing, Function, Teaching experiment

Introduction

First introduced at the elementary grade levels, graphs are essential representations for the study of numerous mathematics topics including modeling relationships between quantities, exploring characteristics of functions, solving for unknown values, and investigating geometric transformations. Highlighting the central role of graphing in mathematics education, the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) contains some form of the term *graph* on more than a third of the document's pages. Building off of the emphasis on graphing at the K-12 level, graphing is central to the study of several undergraduate mathematics courses, including calculus, differential equations, and analysis.

Reflecting the heavy focus on graphing in school mathematics, mathematics education research gives significant attention to graphing, with a number of studies (e.g., Carlson, 1998; Monk, 1992; Oehrtman, Carlson, & Thompson, 2008) having investigated student thinking in the context of the function concept. Although graphing receives a significant focus in mathematics education research, little of this focus includes graphing in the *polar coordinate system* (PCS). The PCS is critical to the study of calculus, complex numbers, and modeling any system that entails radial symmetry or motion about a center point. Despite the important role of the PCS in undergraduate mathematics, available research (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007) suggests students hold limiting understandings of the PCS, where some of their issues stem from problematic connections with the *Cartesian coordinate system* (CCS).

In the present study, we explore connections between student thinking when graphing in the PCS and existing research on student thinking in the context of graphing and function. Specifically, we discuss two undergraduate students' reasoning when graphing functions in the PCS. To graph functions in the PCS, they engaged in several ways of reasoning that ranged from plotting discrete points to reasoning about how quantities continuously vary in tandem. The former way of thinking enabled them to gain a sense of more basic (e.g., constant rate of change) functions, but was not sufficient in and of itself to graph more complex (e.g., trigonometric) functions and connect these graphs to their counterparts in the CCS. By engaging in covariational reasoning, students were more flexibly able to graph relationships in the PCS. Additionally, thinking about functions in terms of covariational relationships enabled the students to conceive graphs in the PCS and CCS as representing the same relationship despite the graphs' visual differences.

Background

The function concept and graphing are widespread in the teaching of mathematics. Yet, research (e.g., Carlson, 1998; Monk, 1992; Oehrtman et al., 2008; Thompson, 1994b) has revealed that students often develop understandings of the function concept, particularly in the context of graphing, that restrict their future learning. For example, students construct function and rate of change understandings that form an impoverished foundation for the study of calculus (Oehrtman et al., 2008; Thompson, 1994b). As Carlson (1998) noted, even high performing calculus students can lack robust understandings of rate of change and exhibit difficulty when interpreting graphs. These difficulties can stem from students' images of function being rooted in visual objects and not two quantities' values varying in tandem (Thompson, 1994b).

Specific to the PCS, Montiel et al. (2008) identified that students' function concept can inhibit their interpretation of graphs in the PCS. Additionally, the authors identified that the (sometimes incorrect) connections students create between the CCS and PCS are often tied to their meanings for functions and graphing in the CCS. For instance, students applied "the vertical line test" to determine if a graphed relationship in the PCS is a function. Similarly, Sayre and Wittman (2007) found that some students rely on the CCS when solving problems better suited for the PCS. Collectively, the studies highlighted that students' ways of thinking about function and graphing often do not support robust connections between the PCS and CCS, nor do their ways of thinking about the CCS support their PCS concept.

Covariational Reasoning and Connecting Coordinate Systems

In exploring students' sense making in the context of graphing and function, Carlson et al. (2002) illustrated the central role of students' capacity to engage in covariational reasoning. *Covariational reasoning*–defined as the "cognitive activities [of an individual] involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354)–is central to students' understanding of numerous precalculus and calculus topics, including exponential functions (Castillo-Garsow, 2010; Confrey & Smith, 1995), trigonometric functions (Moore, 2012), rate of change (Carlson et al., 2002; Thompson, 1994a), function (Oehrtman et al., 2008), and the fundamental theorem of calculus (Thompson, 1994b). While these studies illustrate the importance of covariational reasoning in conceiving relationships between two quantities' values and using the CCS to reflect these relationships, these studies have been limited to the CCS. Due to the importance of covariational reasoning in graphing relationships, we hypothesized that covariational reasoning is critical to students' PCS graphing capabilities.

The mental actions[1] associated with covariational reasoning are not specific to the coordinate system in which one is graphing (nor are they specific to the act of graphing), and thus covariational reasoning characterizes ways of thinking that might support students in connecting relationships represented in multiple coordinate systems. To illustrate, consider

[1] See Carlson et al. (2002) for an elaborate description of the mental actions associated with covariational reasoning.

graphing the relationship defined by $f(x) = x^2$. For the function *f*, $x > 0$, as the *input* increases, the *output* increases with an increasing rate. It follows that the *output* increases such that the *change of output* also increases for successive equal *changes of input*. These *change of output* values increase by a constant amount for successive equal *changes of input*; as the *input changes* from 0 to 1 to 2 to 3 and so on, the *output increases by* 1, 3, 5, and so on; hence, the *change of output* increases by 2 for each successive *change of input* of 1.

The aforementioned covariational relationship can be represented in the CCS ($y = x^2$, Figure 1, left) and PCS ($r = \theta^2$, Figure 1, right). Although changing coordinate systems results in a different visual object, covariational reasoning enables conceiving the graphs in the same way; changing the coordinate system changes the shape of the curve, but the relationship remains invariant. Graphs in different coordinate systems form different visual objects that represent the same relationships because the shape of the graph matters only in that it represents how two quantities' values change in tandem.

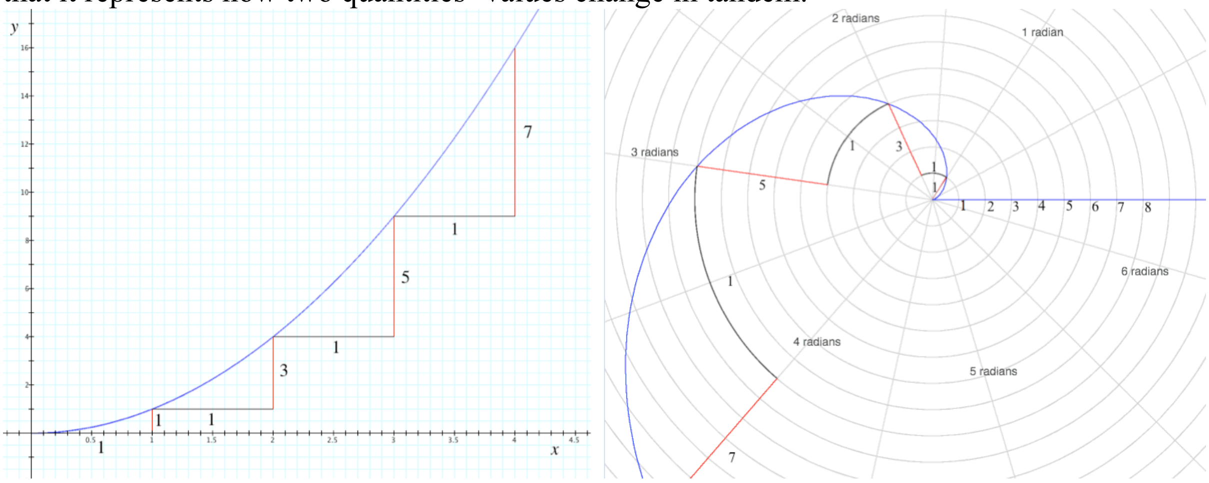

Figure 1. Graphing the same covariational relationship: $y = x^2$ (left) and $r = \theta^2$ (right).

Methodology

Stemming from radical constructivism (Glasersfeld, 1995) underpinnings, which takes the stance that an individual's knowledge is fundamentally unknowable to any other individual, we used qualitative methods to develop *models* of students' thinking (Steffe & Thompson, 2000) that explain their observable behaviors. Specifically, we conducted a teaching experiment (Steffe & Thompson, 2000) to investigate the following research questions:

1. What ways of reasoning do students engage in when graphing functions in the PCS?
2. How do the ways of reasoning identified in the first research question relate to the students' thinking when graphing functions in the CCS?

Subjects and Setting

The subjects of this study (John and Katie) were two undergraduate students enrolled in a pre-service secondary mathematics education program at a large public university in the southeast United States. At the time of data collection, the students were third year (in credits taken) students taking the first pair of courses (one methods and one content) in a pre-service secondary mathematics education program. We chose the students on a voluntary basis from the content course, in which the lead author was the instructor.

The content course engaged the students in quantitative reasoning (Thompson, 1990) and covariational reasoning to explore topics central to secondary mathematics (e.g., trigonometry, exponential functions, linear functions, rate of change, and accumulation). Prior to graphing in the PCS, the course explored ideas of angle measure and trigonometric functions. The approach to these topics was grounded in previous research (Moore, 2012) on students' learning of angle measure and trigonometric functions, and included a significant focus on covariational and quantitative reasoning. We expected the students to be familiar with covariational reasoning when entering the study, but we questioned whether they would or would not spontaneously engage in said reasoning when graphing in the PCS.

Data Collection and Analysis

The teaching experiment (Steffe & Thompson, 2000) consisted of five 75-minute teaching sessions with the pair of students. The first teaching session developed conventions of the PCS (e.g., coordinate pairs representing the distance from a fixed point and the measure of an arc) and supported students' spatial reasoning in the PCS (e.g., considering the location of a point that has a varying arc measure and a constant distance measure, and vice versa). The subsequent teaching experiment sessions, which are the focus of the present report, involved graphing functions of the form $r = f(\theta)$ or $\theta = g(r)$ in the PCS.

The teaching sessions and all student work were videotaped and digitized. Also, fellow researchers observed each teaching session, taking notes of the interactions between the researcher and students. We debriefed immediately after each session in order to discuss the students' thinking and document all instructional decisions. We analyzed the data using an open and axial coding approach (Strauss & Corbin, 1998). The data was first transcribed and instances offering insights into the students' thinking were identified. We then performed a conceptual analysis (Thompson, 2000) of these instances in order to generate and test models of the students' thinking so that these models provided viable explanations of their behaviors. We particularly sought to characterize the students' reasoning when graphing in the PCS and CCS.

Results

After exploring[2] the meaning of coordinates (e.g., a radial distance and an angle measure in radians) and various conventions of the PCS during the first teaching session, the teaching sessions explored representing relationships in the PCS. These relationships included linear functions, quadratic functions, and trigonometric functions. As John and Katie completed the proposed tasks, their solutions offered insights into ways of thinking that support graphing in the PCS and connecting graphs in the PCS and CCS.

Covariational Reasoning and Graphing Relationships

We transitioned into graphing relationships by tasking John and Katie with graphing the function $f(\theta) = 2\theta + 1$. To begin, they graphed the relationship in the CCS (e.g., $y = 2x + 1$) by identifying both the *x*- and *y*-intercepts and connecting these points with a line. The students then graphed the function in the PCS by plotting points for θ values of 0, 1, 2, 3, and 4, and connecting these points (Figure 2). They then continued describing their graphs (Table 1).

Table 1

1	Katie:	They went out by two, like you know here (*pointing at the two in the formula*
2		$r = 2\theta + 1$) the slope is like two (*tapping along the CCS graph*).
3	Int.:	This has no slope (*pointing to the PCS graph*)…
4	Katie:	No, I'm relating the slope here (*pointing to the CCS graph*), to the difference
5		in the radius of two each time (*tapping along the PCS graph*). Like [the
6		radius is] one, three, five, seven, nine, eleven *(pointing to the corresponding*
7		*points on the polar graph)*, [the radius] increases by two.

This interaction illustrates Katie reasoning about the amount of change in the distance

[2] The results of these explorations will be reported elsewhere, but we note that during the construction of the PCS, the students showed little familiarity with the PCS.

from the pole, which she referred to as the "radius," for successive changes of angle measure and connecting this relationship with the "slope" of the line in the CCS. After this interaction, Katie and John claimed that both graphs convey a "constant rate of change" between the input and output values. Katie then added, "That's cool…because you'd never see this (*referring to the PCS graph*) and be like, that's a linear function," suggesting that by conceiving the graphed relationships in terms of covarying quantities' values, they conceived both graphs, which are perceptually different, as representative of the same relationship.

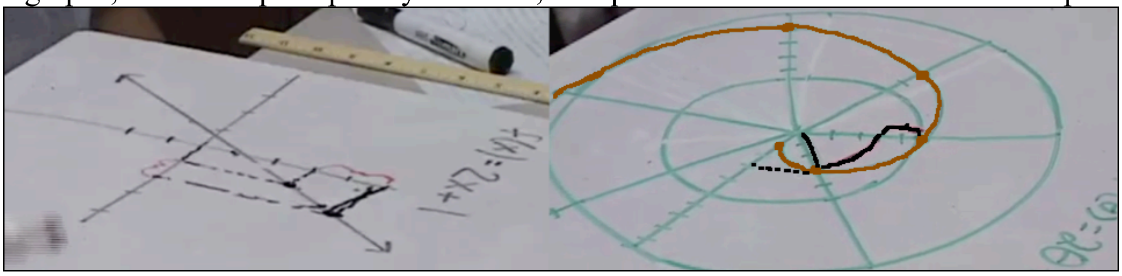

Figure 2. Students' graphs of a constant rate of change relationship.

Based on the students' approach to graphing a linear relationship, we conjectured that similar reasoning would enable them to graph a relationship with a non-constant rate of change. We asked the students to graph the relationship $r = \theta^2$, which they spontaneously compared to the relationship $r = \theta$. Like their solution to graphing the linear function, the students first plotted points and connected the points. They then compared their graphs (see Figure 3 for their written products) (Table 2).

Table 2

Katie:	r of theta, but compared to r of theta squared it's like expanded (*Katie points to the two graphs and then spreads her hands apart*). Like, like, this one's like much more tighter swirled (*moving her hands in a circular motion*) but then this one (*referring to quadratic*) is just like looser I guess.
John:	Yeah, we can see better, with both of them, both graphs, that the change in radius (*referring to quadratic*) for every radian further that the angle is increasing (*rotating his hand in successive movements while spreading his index and middle finger apart*)…Um, the radius, every time is increasing at an increasing rate (*referring to quadratic*).
Int.:	Okay now what's that mean in terms of amounts of change?
John:	We could do equal changes in theta and then…
Katie:	Like, if we looked at first these two then these two points (*indicating the points (9, 3) to (16, 4), and then (16, 4) to (25, 5)*), the change of theta here would be this, that length (*drawing an arc from (9, 3) to (9, 4)*). But then the change is radius would be up that line (*drawing a segment from the point (9, 4) to (16, 4)*).
John:	Which is seven.
Katie:	And then we have the same thing (*draws an arc from (16, 4) to (16, 5) and a segment from (16, 5) to (25, 5)*)…so you can see these black lines, the [change in radius] is increasing.
John:	So that's like nine to sixteen (*pointing to the segment connecting the points (9, 4) and (16, 4)*), which is seven, and this one is sixteen to twenty-five (*pointing to the segment connecting the points (16, 5) and (25, 5)*), which is nine, which we can see there too (*pointing to the Cartesian graph*).

Katie first compared the perceptual shapes of the graphs (lines 1-4). Following this, the students reasoned about amounts of change and rates of change between the two quantities to compare and make sense of each graph's shape. Specifically, the students reasoned that the graph of $r = \theta^2$ is "looser" or moves away from the pole "faster" because *r* increases at an increasing rate with respect to an increasing θ, which they confirmed by identifying specific changes in the quantities' values.

Immediately following John's last statement (line 24), the students denoted amounts of change on a graph in the CCS (Figure 3), while Katie claimed, "Like our change input here (*referring to CCS graph*) would represent the change in this angle measure (*indicating the corresponding change on the PCS graph*), and then our output, change of radius length, and that's increasing for equal changes." Thus, compatible with the students' actions when graphing the linear function, the students constructed a structure of covarying quantities that enabled them to see a graph in the PCS and a graph in the CCS as one in the same.

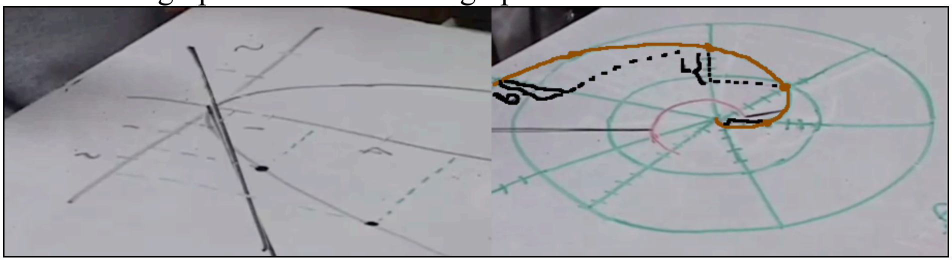

Figure 3. Students' graphs of the quadratic function.

Covariational Reasoning and Conceiving a Trigonometric Function

Over the course of the study, Katie and John leveraged covariational reasoning to construct and connect graphs in both coordinate systems. We thought the students might engage in similar reasoning to interpret a *given* graph in the PCS, and thus we tasked the students with determining a formula for a given graph ($r = \sin(\theta)$, Figure 4). After identifying *r* values corresponding to θ values of 0, π/2, π, and 3π/2, the students conjectured that $r = \sin(\theta)$ is the appropriate formula for the given graph. The students then drew a graph of the sine function in the CCS and explained their solution (Table 3).

Table 3

Katie:	So we start here *(pointing to the pole in the PCS).*
John:	Ya, and we're sweeping around *(making a circular motion with his hands).* As theta's increasing, distance away from the origin is increasing *(Katie traces along the polar graph from 0 radians to π/2 radians)* and then decreases again…it increases until pi-over-two and then it starts decreasing.
Int.:	And then what happens from like pi to two-pi?
Katie:	It's the same.
John:	Same idea except, the radius is going to be negative, so it gets more in the negative direction of the angle we're sweeping (*using marker to sweep out a ray from π to 3π/2 radians – see Figure 3*) until three-pi-over-two, where it's negative one away.
Katie:	This is the biggest in magnitude, so it's the furthest away (*placing fingers at (1, 3π/2) and (1, π/2)*), and then [the distance] gets smaller in magnitude (*tracing one index finger along an arc from (1, 3π/2) to (1, 2π) and the other index finger along the graph – see Figure 3*).

When making sense of the graph, and compatible with the previous interaction (Tables 1-2), the students used a combination of identifying points and covariational reasoning. Specifically, the students reasoned about the distance from the pole as increasing or decreasing for an increasing angle measure to make sense of the relationship conveyed by the graph. For instance, Katie reasoned that as the angle measure increases from 3π/2 radians to 2π radians, the distance from the pole decreases from a magnitude of one to zero, which corresponds to the value increasing from -1 to 0 (Figure 4). Following this interaction, the students continued justifying their formula by describing that a graph of $y=\sin(x)$ in the CCS conveys the same covariational relationship with identical critical points as the PCS graph. Thus, like the previous tasks, by identifying various points and engaging in covariational reasoning, the students identified the formula for the given graph and concluded that both the CCS and PCS representations of the formula convey the same relationship.

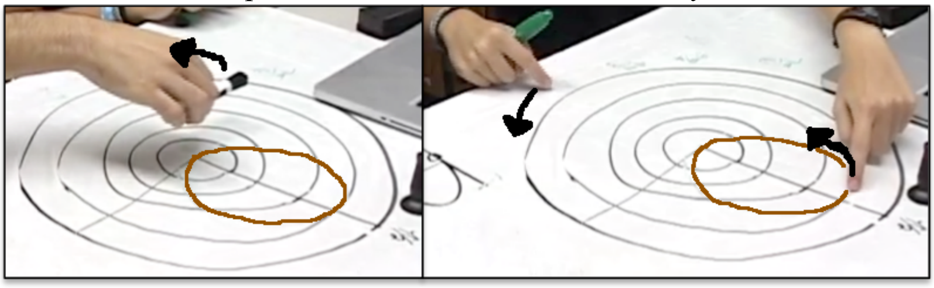

Figure 4. Students covarying quantities.

Discussion and Implications

In the present study, the students spontaneously reasoned about quantities' values varying in tandem when graphing and interpreting functions in both the CCS and PCS. The students' capacity to reason about graphs as conveying covariational relationships appeared to support connections among graphing functions in both systems. That is, as the students moved from coordinate system to coordinate system, covariational reasoning enabled the students to understand each representation (including the formula) as conveying the same relationship despite differences in the visual features of the graphs (and formula). Such reasoning, in combination with understanding the conventions of the PCS, might support avoiding the difficulties that Montiel et al. (2008) found when working with calculus students.

The findings of this study suggest that a *potential* benefit of incorporating the PCS in the study of mathematics is increasing an emphasis on reasoning that enables a student to approach graphs in both systems in compatible ways. Investigating graphing only in the CCS has the possible consequence of reinforcing common student conceptions of the function concept and graphing that do not entail reasoning about covarying quantities (e.g., conceiving graphs as pictures). By prompting students to transition from coordinate system to coordinate system, a need can be established for ways of thinking (e.g., covariational reasoning) that enable conceiving graphs in each system as conveying the same relationship. As such, graphing in both systems might foster abstractions stemming from various operations involved in covariational reasoning (e.g., rate of change reasoning and coordinating amounts of change). For instance, for Katie, graphing in both coordinate systems seemed to foreground the "constant rate of change" of a linear relationship (Table 1), as opposed to the slope of a line. At this time, this potential use of the PCS in secondary mathematics remains merely a hypothesis, and future studies should investigate promoting covariational reasoning through the use of the PCS in combination with the CCS.

We also note that the students appeared to engage in a combination of *smooth* and *chunky* images of change, as defined by Castillo-Garsow, Johnson, and Moore (submitted), when graphing relationships in the PCS. Throughout the instructional sequence, the students relied on first graphing discrete points and comparing discrete amounts of change between these points, which is suggestive of chunky images of change. The students also exhibited behaviors consistent with smooth images of change. For instance, the students reasoned about relationships in terms of one quantity increasing or decreasing for a continuous increase in the other quantity to make sense of graphs' behaviors between identified points (e.g., Table 3). The students' actions highlight the importance of both images of change for graphing relationships and constructing connections from one coordinate system to another, suggesting that instruction should emphasize both ways of reasoning so that they work in tandem. As the authors (Castillo-Garsow et al., submitted) described, these ways of thinking have different mathematical roots and consequences, and additional research is needed to investigate relationships between these ways of thinking in the context of student learning.

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