PRE-SERVICE SECONDARY TEACHERS' MEANINGS FOR FRACTIONS AND DIVISION

Cameron Byerley, Arizona State University
Neil Hatfield, Arizona State University

In this study, seventeen math education majors completed a test on fractions and quotient. From this group, one above-average calculus student was selected to participate in a six-lesson teaching experiment. The major question investigated was "what constrains and affords the development of the productive meanings for division and fractions articulated by Thompson and Saldanha (2003)?" The student's thinking was described using Steffe and Olive's (2010) models of fractional knowledge. The report focuses on the student's part-whole meaning for fractions and her difficulty assimilating instruction on partitive meanings for quotient. Her part-whole meaning for fractions led to the resilient belief that any partition of a length of size m must result in m, unit size pieces. It was non-trivial to develop the basic meanings underlying the concept of rate of change, even with a future math teacher who passed calculus.

Key Words: Pre-service Secondary Teachers, Rate of Change, Fractions, Teaching Experiment, Division

Research indicates that many students leave the middle grades well trained to operate symbolically on fractions while having under-developed meanings for fraction symbols (Hiebert & Behr, 1991). It has been suggested that strong meanings for secondary topics such as rate of change and proportion are dependent on strong meanings for fractions (Norton & Hackenberg, 2010). Although there are not many studies connecting students' meanings of fractions to their meanings for rate of change, the potential relationship can be justified mathematically. Rate of change can be understood as a comparison of the relative sizes of associated changes in two quantities. The mature meaning of fractions as reciprocal relationships of relative size, described by Thompson and Saldanha (2003), coheres well with the above meaning of rate of change. Research makes it clear that secondary teachers can anticipate needing to help their students develop quantitative meanings for fractions. Furthermore, many studies report that rate of change is a challenging topic for high school and university students (Asiala, Dubinsky, Cottrill, & Schwingendorf, 1997; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Coe, 2007; Orton, 1983). With the acknowledgment that we have no evidence demonstrating that immature meanings for rate of change are related to immature meanings for fractions, we propose that it is important to study secondary teachers' meanings for fractions because of the mathematical importance of comparisons of relative size in secondary mathematics. Furthermore, secondary teachers will be more likely to address their student's weak meanings for fractions if they have a strong quantitative understanding of the topic.

Most, but not all, studies designed to investigate teachers' meanings for fractions or division describe elementary teachers. Numerous researchers have found that teachers find it difficult to model a situation using division (Ball, 1990; Simon, 1993).

Additionally, teachers' demonstrations of fractions and division primarily involve computations and do not focus on quantitative meanings for the operations (Harel & Behr, 1995). Finally, teachers struggle to draw models of fraction situations while teaching (Izsák, 2008). In prior attempts to influence teachers' fraction schemes it was found "challenging at best and impossible at worst" to encourage teachers to override automatized procedures for computation and think meaningfully about multiplying and dividing fractions (Armstrong & Bezuk, 1995).

Methodology and Research Question

To understand the challenges in teaching adults new meanings for fractions we conducted a six session teaching experiment with a pre-service secondary mathematics teacher named Jacqui (Steffe & Thompson, 2000). The instructional goal was to develop the reciprocal relationships of relative size meaning for fractions articulated by Thompson and Saldanha (2003) by engaging the pre-service teacher in a modified activity sequence originally designed by O'Bryan and Thompson (unpublished). During the teaching experiment the witness, second author, and the instructor, first author, assumed that Jacqui had different meanings for fractions and division than we did. Cobb and Steffe (2011) describe a student's "actual meaning" as the one that is created as the student interprets instruction and not necessarily the teacher's intended meaning (p. 86). We believed that by engaging her in specific tasks her thinking could be modeled in a way that could shed light on the challenges associated with teaching adults new meanings for fractions. The teaching experiment was videotaped, transcribed and analyzed using methods described by Corbin and Strauss (2007). The primary research question was "what meanings for fractions and division constrain and afford the development of a reciprocal relationships of relative size meaning for fractions?

Constructs Used to Model Meanings for Fractions

Descriptions of Jacqui's thinking are based on the constructs reciprocal relationships of relative size from Thompson and Saldanha (2003), part-whole and iterative fraction schemes from Steffe & Olive (2010), and partitive and quotitive division described by Simon (1993).

Thompson and Saldanha (2003) stress that a multiplicative understanding of fraction requires that students understand fractions as reciprocal relationships of relative size. They describe this relationship as "Amount A is $1/n$ the size of amount B means that amount B is n times as large as amount A. Amount A being n times as large as amount B means that amount B is $1/n$ as large as amount A" (2003, p. 31). Thompson and Saldanha contrast this mature multiplicative meaning with an additive part-whole fraction scheme.

In a part-whole fraction scheme the student understands seven-tenths as seven out of ten equal parts without an awareness that the one-tenth is an iterable unit or a length that stands in comparison to the whole (Steffe & Olive, 2010, p. 323). For example, Nik Pa (1987) found that 10 and 11 year old children could not find 1/5 of 10 items because "one-fifth" referred to one and five single items (cited in Steffe and Olive, 2010, p. 3). A part-whole meaning is often sufficient to correctly answer common fraction problems such as describing what fraction of marbles in a bag are red. Notice that in the statement "3/5 of the marbles are red" the denominator refers to the number of marbles in the bag.

In the more advanced iterative fraction scheme, the denominator refers to the number of times the original quantity is partitioned, and not necessarily the number of distinct objects in the physical situation. Students with an iterative fraction scheme can imagine fractions such as 7/5 as a length because they can imagine disembedding 1/5 of a whole and iterating it seven times. After a student partitions, disembeds, and iterates to find 7/5 the result stands in "multiplicative relationship to the whole" meaning that it is 1/5 seven times (Hackenberg, 2010, p. 394). Although Steffe and Olive (2010) and Thompson and Saldanha (2003) do not explicitly draw connections between their constructs, it seems mathematically logical that developing a reciprocal relationships of relative size meaning for fractions requires an iterative fraction scheme and an ability to reverse mental operations.

Understanding fractions as reciprocal relationships of relative size requires a quantitative meaning for the quotient. There are at least three quantitative meanings that allow one to interpret the quotient as providing information about real-world quantities. In the partitive meaning the quotient A/B refers to the size of each group when A is cut up into B equal-sized groups. In the quotitive meaning, the quotient A/B represents how many times something of length B would fit when laid end to end next to something of length A (Simon, 1993). Quotitive division could also be described as measuring A with a ruler of length B. In the third meaning, A/B quotient A/B tells us the relative size of quantity A and B. In the relative size meaning A is A/B times as large as B (Thompson & Saldanha, 2003). Speaking of division as a measure of relative size is particularly useful when trying to explain rate of change or the definition of the derivative.

Results of Initial Assessment and Teaching Experiment

Jacqui was selected as a typical representative of a group of 17 pre-service secondary mathematics teachers who completed an assessment on meanings for division and fractions. Assessment questions were inspired by Hackenberg's (2009) research on reversible multiplicative relationships, Simon's (1993) and Ball's (1990) studies on division, O'Bryan's and Thompson's activity sequence and Coe's study of teachers' meanings for rate of change (2007). Jacqui received A's and B's in all of her math courses up to and including Calculus I and plans to teach high school math. Although everyone who took the assessment had at least passed Calculus I, the majority struggled to answer questions related to fractions and division. Ten out of seventeen pre-service teachers' were able to give a scenario in which you would divide by a fraction. Six out of seventeen were able to use a picture to explain the meaning of a quotient in a problem involving division by a decimal. Only one teacher was able to explain why division is used to calculate slope. The acceptable response described division as a measure of relative size in changes in y and changes in x. Like many respondents, Jacqui was able to correctly answer some tasks, and drew visual representations of fractions, but struggled to explain or model division and tended to express fractions as parts out of wholes.

After a summary of the teaching experiment, two of Jacqui's resilient, problematic meanings will be described in more depth. Throughout the teaching experiment the witness and instructor attempted to help Jacqui develop an iterative fraction scheme by interpreting A/B as A copies of $1/B$ of one. This required developing a meaning for multiplication as making copies as well as understanding $1/B$ as the amount in one piece when we partition one into B equal pieces. The item in *Figure 1* was intended to help Jacqui interpret fractions as reciprocal relationships of relative size.

Some amount, call it B, is partitioned into n equal parts.

a) How large is B compared to the size of each part?

b) How large is each part in relation to B?

Figure 1. Teaching experiment item designed by O'Byran and Thompson.

Figure 2 shows an item added from Hackenberg's research in an attempt to necessitate the coordination of multiple levels of units, an ability thought to be related to advanced fraction schemes (Steffe & Olive, 2010).

The unmarked rectangle shown represents 3/5ths of a candy bar.

a) Draw a picture of the whole candy bar.

b) Suppose the entire bar is shared equally among 10 students. What fraction of the entire bar will they receive? Why?

c) What fraction of 1/5th of the bar will they receive?

Figure 2. Teaching experiment item inspired by Hackenberg (2010).

Additionally, we wanted Jacqui to understand that an iterative fraction scheme could give a quantitative meaning to an "improper" fraction because in her initial assessment she interpreted improper fractions only after converting them to mixed numbers. Although Jacqui was often able to answer the questions correctly using procedures and her primarily part-whole meaning for fractions, she revealed many problematic assumptions when she explained her work. For example, she believed that fractions must be less than one, that multiplication makes bigger, and that the word "of" in "1/4 of 1/6" means division because in her diagram it looked like she was "pulling out" 1/6 from the whole 24 times. She did not think that dividing by a fraction had the same quantitative meaning as dividing by a whole number because dividing by a fraction was really multiplying. When asked how fractions and division were related she replied that it was possible to divide by a fraction but gave no evidence of a stronger connection between the two ideas. After these issues were noticed, it was typically possible to ask Jacqui a question that caused her to see her mistake. With practice, she learned to speak of A/B as A copies of $1/B$, the product $u*v$ as u copies of size v and to use division to determine how many times as large A is as B, where A and B were any real numbers.

Despite some success, there were two related issues that took a number of focused attempts to resolve that were strongly rooted in Jacqui's meanings for division and fractions. The first issue was that we wanted Jacqui to distinguish between partitive and quotitive division so that when she spoke about dividing while explaining her answers we had an image of what she meant. On the first day we gave an example of the difference between the partitive and quotitive division and asked her to summarize the two models at the end of the day. In both of her models the quotient A/B represented the number of groups of size B when A was measured with length B. She used slightly different language to explain each model. In one case she described seeing how many times B could be pulled out of A. In the other case she described how many copies of B fit into A. In these explanations, Jacqui used the word partition to describe a quotitive model of division. She said "I'm trying to figure out how many times one third can go into four thirds so I can partition four thirds into one third sections and then evaluate how many one third pieces make up a complete four thirds piece." We immediately made the first of many attempts to check that Jacqui understood our meaning for partition by asking her to interpret the quotient as the size of the group resulting from a partition. In the next session we asked her to summarize the two models of division and she again explained two quotitive models. We intervened with another explanation of the partitive model of division and asked her to explain what A/B meant in the partitive model. She replied, "so you are partitioning A into B pieces… So how ever many times, so each one is $1/B$ of A, so [pause] I don't know where to go from here. That's just it, it's just $1/B$ of A." Despite direct instruction moments before, she had not assimilated that the size of the piece resulting from the partition stands for the quotient. We attempted to resolve this problem on five separate occasions and after a 25-minute focused lesson on partitioning using manipulatives she still had a different meaning for partitioning that resembled the quotitive model.

The other problematic issue for Jacqui was her repeated insistence that one part out of a length A cut up into B equal pieces was size $1/A$. She also divided a length of four into two equal pieces and called the size of each piece $1/4$ instead of $1/2$. She confounded the total number of objects in a group (four) and the number of partitions of the group (two). Her behavior is consistent with Steffe's description of part-whole meanings for fractions that focus on the denominator as the total number of objects in a group. This meaning for fractions is problematic when attempting to understand the statement "Amount A is $1/n$ the size of amount B means that amount B is n times as large as amount A." In Jacqui's case she automatically assumed that after cutting up a quantity into groups that each group must be size one. Jacqui struggled with the question in *Figure 3* because of this assumption.

Imagine that a pack of bubble gum is split equally among a group of 11 friends. What fraction of the bubble gum will each friend receive?

Figure 3. Teaching experiment item from O'Bryan and Thompson.

Although she knew that each person received 1/11 of the pack she repeatedly insisted that there must be 11 wrapped pieces of gum in the pack and that if the pack was partitioned equally it was not possible for one person to receive 1/2 of a wrapped piece of gum. Even after a prolonged discussion about the bubble gum problem, her part-whole meaning fractions took over and she continued assume the denominator represented the total number of objects in the next problem. Another indication of her part-whole thinking was her tendency to describe the fraction 5/5 as "five pieces of five" because "you look at it as five pieces of length one."

Although Jacqui initially viewed fractions and division separately, she began to associate both fractions and division with the expression A / B. It appears that Jacqui knew a process for finding a quotient using the quotitive method, but when she imagined partitioning an amount into equal pieces she automatically assumed that he pieces were of size one. This is one explanation for why she would not have assimilated the instruction focused on interpreting the quotient as the size of a group resulting from a partition. Furthermore, her meaning for the word partition was associated with the hash marks she made when she measured A in terms of length B. When I referred to the partitioning model of division, she imagined quotitive division and learned a meaning for partition that was inconsistent with my intentions.

Jacqui's Constraints and Affordances

Speaking carefully, anticipating and checking for non-productive meanings and watching videotaped sessions all contributed to the resolution of a number of issues Jacqui faced. Many of the cases of miscommunication were subtle and only apparent in retrospect. It became clear that drawing a rectangle partitioned into pieces could mean partitioning to one person, copying to another and measuring to a third. A major constraint was that Jacqui believed she understood me, when in fact she had assimilated what I had said to unintended images. She openly admitted she believed we were just using different words but speaking about the same idea when we were in the midst of a major miscommunication about partitive and quotitive division. It is possible that when teachers are asked to learn a new language about fractions and they do not develop the associated quantitative meanings, they will view the request as arbitrary pickiness.

Another constraint to developing the intended meanings is that Jacqui often was able to provide descriptions such as "9/7 means nine copies of 1/7" without altering her problematic assumptions about the size of the part in a fraction. Using symbolic skills paired with part-whole meanings for fractions, Jacqui was able to answer questions that were designed to challenge students who had primarily part-whole meanings in Hackenberg's research (2010). Often she inserted correct numbers in the blanks in the activity sequence and needed personal feedback on the quality of her explanations to alert her to errors in her thinking. Even after substantial intervention, we did not reach our goal of developing a reciprocal relationships of relative size meaning with Jacqui. The part-whole fraction meaning Jacqui most likely developed in school constrained her ability to assimilate partitive division. This lack of shared meaning for the word partition was one major constraint that stalled our progress in developing a reciprocal relationships of relative size meaning for fractions. Although not formally investigated in this study, it seems possible that Jacqui's difficulty explaining the slope formula could be because she struggled to view division and fractions as a measure of relative size of two quantities.

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