NOT ALL INFORMAL REPRESENTATIONS ARE CREATED EQUAL

Kristen Lew, Juan Pablo Mejia-Ramos, Keith Weber
Rutgers University

Some mathematics educators and mathematicians have suggested that students should base their proofs on informal reasoning (Garuti et al. 1998). However, the ways in which students implement informal representations are not well understood. In this study, we investigate informal representations made by undergraduates during proof construction. Their use of informal representations will be compared to mathematicians' use of informal representations as described in Alcock (2004) and Samkoff et al. (2012). Further, an analysis of different types of informal representations will investigate the necessity to treat these different representations more carefully in the future.

Key words: Informal representations, proof, proof construction

Mathematicians use informal representations, including reasoning from graphs, diagrams, and specific examples of more general concepts, to guide their proof construction (Thurston, 1994; Burton, 2004; Hadamard, 1945). For this reason, mathematics educators argue that students should base their proofs off of informal representations as well (Garuti et al. 1998). However, mathematics educators have also documented that students are not always successful when basing their proofs off of informal representations (Alcock & Weber, 2010; Pedemonte, 2007). In order to learn what distinguishes students who are successful when using informal representations to guide their proof production from those who are not, students' attempts to use these informal representations must be studied. This study is intended to address the need to better understand how students use informal representations in their proof productions.

Theoretical Perspective

We define a representation in mathematics to be a visual or algebraic portrayal of a concept or a situation. A formal representation would consist of a formal, rigorous definition, whereas an informal representation is any representation that is not formal. As such, example objects and visual representations of mathematical concepts are both considered informal representations in this study.

Garuti et al. (1998) argue that students' proofs should be based on informal reasoning, describing a cognitive unity that should exist between the way a student comes to informally understand the veracity of a theorem and the formal proof. This is consistent with Raman (2002) and Weber & Alcock (2004), who suggest that it is desirable for students' proofs be based on informal representations. In order to investigate students' use of informal representations, we compare students' proof productions against the ways in which mathematicians use informal representations based on the typologies of Alcock (2004) and Samkoff et al (2012).

Alcock (2004) addresses the ways in which mathematicians use example objects in producing a proof. Mathematicians were found to use examples to understand statements by (1) instantiating example objects, (2) generate arguments both directly and indirectly, and (3) check arguments by considering counterexamples. Alcock (2004) describes how mathematicians generate arguments directly by "arguing about or manipulating a specific example and translating this to a general case" and indirectly by "trying to construct a counterexample and attending to why this is impossible" (p. 21).

Samkoff et al. (2012) noted the ways in which mathematicians used diagrams in producing a proof. Mathematicians used diagrams to (1) notice properties and generate conjectures, (2) estimate the truth of an assertion, (3) suggest a proof approach, (4) instantiate or represent an idea or assertion in a diagram, and (5) verify the theorem using the diagram.

Research Questions

Within the framework of this study, which suggests that informal representations should be used in proof production, a stronger understanding of how students use informal representations can help mathematics educators to understand why students are sometimes unsuccessful in their proof mathematics courses. We hope to address the following questions:

1. Do students use informal representations, including examples and diagrams, for the same purposes that mathematicians do (as claimed by Alcock, 2004, and Samkoff et al, 2012)?
2. Do different types of informal representation lead to different uses?
    a. Do visual representations lead to different uses than algebraic representations?
    b. Do representations of specific mathematical objects lead to different uses than representations of general mathematical objects?

We note that this study is in the scope of linear algebra and calculus. As such the results of our analysis may not be generalizable to all areas of mathematics.

Methods

Data Collection

Participants were mathematics majors from a large public university in the northeastern United States who had completed their mathematics requirements for graduation. Data was collected in form of individual task-based interviews. We designed the tasks such that they could be approached both syntactically and semantically, so that informal representations had the opportunity to play a significant role in proof production. Students completed 14 proof tasks (of varying difficulty with 7 in calculus and 7 in linear algebra) during two meetings. Students were asked to think aloud and were given ten minutes for each task. Definitions and examples for relevant concepts and computer graphing software were accessible to the participants. From these interviews we have both the video data and the student work.

Analysis

Coding. Using the data, we identified over 270 informal representations. Each informal representation is analyzed for subsequent actions that followed the construction of a representation. Such actions are coded as one of the following: making an inference, giving an explanation, modifying a visual representation, verifying the theorem, verifying a statement (that is not the theorem), suggesting a proof approach, constructing counterexamples, or using a counterexample strategy. These actions are described in further detail in the Appendix. Actions were coded only if they were either immediately located or had explicit reference to the informal representation.

Question 1: Comparing mathematicians and students. In order to investigate this research question, we analyze students' use of informal algebraic representations in relation to Alcock (2004) and use of informal visual representation in relation to Samkoff et al. (2012).

Use of informal algebraic representations. To investigate students' use of informal algebraic representations, we employed the typology as described in Alcock (2004). To affirm that students and mathematicians use informal algebraic representations for the same purposes, we would expect to find students using informal representations to understand statements, generate arguments, and check arguments. For evidence of understanding statements, we would expect to find codes of verifying statements that are not the theorem to be proved; for instance, verifying particular concepts in the given statement in the proof, statements made by the student, and definitions are examples of understanding statements.

Next, in generating arguments there are two situations as described by Alcock (2004). First, students directly generating arguments would entail that we find codes of giving explanations. Second, students indirectly generating statements would suggest that we find counterexample strategy codes. Finally, to find evidence of students using informal representations to check arguments, we would expect to find codes of students constructing counterexamples with the intent of checking an argument.

Use of informal visual representations. To investigate students' use of informal visual representations, we employ the typology described in Samkoff et al. (2012). To affirm that students and mathematicians use informal visual representations for the same purposes, we would expect to find: a) inference codes as evidence of students noticing properties and generating conjectures, b) verifying statement codes to show students estimating the truth of an assertion, c) students suggesting a proof approaches, d) modification of visual representations as evidence of students instantiating or representing an idea or assertion in the diagram, and d) codes of verifying the statement to show students validating the theorem using a diagram.

Question 2: Different types of informal representations. During our analysis, we noticed that there were four types of informal representations, as exemplified in Table 1. Informal representations were classified not only as specific or general, but also as visual or algebraic.

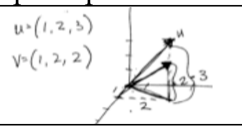
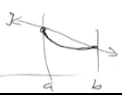
	Sample representation	Description
Specific-Visual		Graphs and diagrams representing specific situations/objects.
Specific-Algebraic	$\begin{pmatrix}0\\0\\2\end{pmatrix} \in W$	Algebraic representations using numbers to exhibit specific situations/objects.
General-Visual		Graphs and diagrams representing classes of situations or general objects.
General-Algebraic	$f(x)=x^{2n}$ $f'(x)=2nx^{2n-1}$	Algebraic representations using variables to exhibit classes of situations or general objects.

Table 1. Four types of informal representations.

In order to investigate whether different types of informal representations lead to different uses, we will analyze the total counts of each type of action code that follows each type of informal representation (specific visual, specific algebraic, general visual, and general algebraic). Considering the percentages of the occurrences of each action code per type of informal representations, we would expect there to be differences in the percentages to affirm that different informal representations lead to different uses.

Discussion

This study is in its preliminary stages and analysis has not yet been completed. However, we have coded one quarter of the data and have some tentative preliminary results. We have thus far not found evidence for students using informal algebraic representations to generate arguments or to check arguments in the sense of Alcock (2004). In contrast, we have found evidence that students use informal visual representations for purposes similar to the mathematicians in Samkoff et al. (2012).

Further, our analysis to date also suggests that students do use specific visual, specific algebraic, general visual, and general algebraic informal representations for different uses. For example, students so far appear to use specific algebraic representations to understand statements more frequently than they do with generic ones. Students also appear to use specific visual representations to estimate the truth of an assertion, whereas they have not done so with generic visual representations in our analysis so far.

These results may shed some light on why some students are unsuccessful in their proof productions. Thus far, it appears that students may use visual representations in the same manner that mathematicians do, but that this is not so with algebraic examples. If students are not in the habit of generating and checking arguments using informal algebraic representations, as our analysis to date suggests, they can perhaps improve their proof construction if they can be encouraged to do. However, these results are preliminary so we make no claims of generality.

If these trends continue throughout the analysis, then these results would suggest that more attention should be paid to the instruction of students' use of algebraic representations. Moreover, the result that students use visual representations in ways that are similar to mathematicians would lead to further questions. For example, if students and mathematicians use visual representations in their proof productions in similar ways, why do students continue to struggle in their proof productions?

Next, if the analysis on the different types of representations continues to show that these four different types of representations lead to different purposes, the results would highlight the need to treat these informal representations differently – both in future studies and in presentation to students.

Questions for the Audience

Are there other frameworks for mathematicians' use of examples or diagrams that we did not consider? Are there other ways to code our data?

References

Alcock, L.J. (2004). Uses of example objects in proving. In PME 28, Bergen, Norway.

Alcock, L.J. and Weber, K. (2010). Undergraduates' example use in proof production: Purposes and effectiveness. Investigations in Mathematical Learning 31(1), 1-22.

Burton, L. (2004). Mathematicians as enquirers: Learning about learning mathematics. Berlin: Springer.

Garuti R., Boero P. & Lemut E. (1998). Cognitive unity of theorems and difficulty of proof. Proceedings of the 22nd PME Conference, Stellenbosh, South Africa, 2, 345 – 352.

Hadamard, J. (1945). The Psychology of invention in the mathematical field. Princeton, NJ: Princeton University Press.

Raman, M. (2002). Coordinating informal and formal aspects of mathematics: Student behavior and textbook messages. Journal of Mathematical Behavior, 21, 135–150.

Thurston, W.P. (1995). On proof and progress in mathematics. For the Learning of Mathematics, 15(1), 29-37.

Weber, K. and Alcock, L.J. (2004). Semantic and syntactic proof productions. Educational Studies in Mathematics, 56(3), 209-34.

Appendix

Description of Actions that Follow Informal Representations

Action	Description of Action
Making an inference	A participant notices something that is true about an example or diagram that was not yet under consideration.
Giving an explanation	A participant mentions what he believes to be general properties of the informal representation that make a statement true.
Modifying a visual representation	A participant modifies or transforms an informal visual representation that has been considered previously.
Verifying the theorem	A participant constructs a graph or example and explicitly comments that he/she thinks the theorem to be proved is or is not true.
Verifying a statement	A participant constructs a graph or example and explicitly comments that he/she thinks a statement (not the theorem to be proved) is or is not true.
Suggesting a proof approach	A participant notes a proof approached based on the informal representation.
Constructing counterexamples	A participant constructs counterexamples for the purpose of verifying a statement or an argument.
Using a counterexample strategy	A participant attempts to construct counterexamples for the purpose of seeing why such counterexamples cannot exist.