DEVELOPING FACILITY WITH SETS OF OUTCOMES BY SOLVING SMALLER, SIMLER COUNTING PROBLEMS

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Combinatorial enumeration has a variety of important applications, but there is much evidence indicating that students struggle with solving counting problems. In this paper, the use of the problem-solving heuristic of solving smaller, similar problems is tied to students' facility with sets of outcomes. Drawing upon student data from clinical interviews in which post-secondary students solved counting problems, evidence is given for how numerical reduction of parameters can allow for a more concrete grasp of outcomes. The case is made that the strategy is particularly useful within the area of combinatorics, and avenues for further research are discussed.

Keywords: Counting, Combinatorics, Discrete Mathematics, Problem Solving

Combinatorial topics have relevant applications in areas such as computer science and probability (e.g., Jones, 2005; Polaki, 2005), and they provide worthwhile contexts in which students can engage in meaningful problem solving. In his undergraduate textbook *Applied Combinatorics*, Tucker (2002) says of his counting chapter, "We discuss counting problems for which no specific theory exists" (p. 169), emphasizing combinatorics as an ideal setting for fostering meaningful problem solving and rich mathematical thinking. Research indicates, however, that students face difficulties with combinatorial concepts, and this is certainly true at the undergraduate level (e.g., Lockwood, 2011a; Eizenberg & Zaslavsky, 2004; Godino et al., 2005; Hadar & Hadass, 1981). Given such difficulty, there is a genuine need for researchers to identify specific areas of struggle for students and to attend to potential ways in which students may improve in their combinatorial problem solving.

In this paper, the emphasis is focused on one particular aspect of combinatorial problem solving – the use of the problem solving heuristic of solving smaller, similar problems. The research goal addressed in this paper is to highlight the value of this heuristic in facilitating students' use of sets of outcomes. The case is made that work with smaller problems is a significant problem solving strategy for combinatorial tasks, particularly because such work can produce facility with sets of outcomes.

To clarify what is meant by "smaller, similar problems" in this paper, consider the following. In a counting problem, there are typically a number of conditions that specify what the problem is asking. These conditions refer to the rules or limitations that must be met in a given problem. Some of these might be numerical in nature (e.g., the specific number of letters in a password), but others might refer to non-numerical conditions (e.g., the fact that repetition of letters is allowed in a password). For clarity, numerical conditions are referred to as *parameters,* and non-numerical conditions are referred to as *constraints*. In relation to a given problem, any other problem that a student might attempt to solve is called "smaller" if it reduces one or more of the parameters in some way, and is called "similar" if it generally maintains the constraints of the original problem.

Literature Review and Theoretical Perspective

When talking about reducing problems to smaller, similar problems, an important issue is whether the similarity exists in the eyes of the student or the researcher. In an earlier paper, Lockwood (2011b) suggests *actor-oriented transfer* (AOT) as a valuable lens through which to examine students' combinatorial problem solving. Lobato (2003) introduced the notion of AOT, a methodological perspective in which the researcher focuses on student-generated connections between problems, and not on connections that the researcher may expect. In this study, in line with Lobato (2003), student-generated similarities are emphasized, not relationships that the researcher determined to be similar.

The specific strategy of solving smaller related problems has been alluded to by problem solving researchers like Schoenfeld (1979, 1980), Polya (1945), and Silver (1979, 1981). Polya discusses this strategy in terms of "discovering a *simpler analogous problem*" (p. 38, emphasis in original) and suggests that solving such a problem provides a model to follow when solving the original problem. Schoenfeld (1979) conducted a study examining the effectiveness of explicitly teaching problem solving heuristics, and considering "a smaller problem with fewer variables" (p. 178) was one such heuristic that he examined. Schoenfeld's and Polya's attention to such a strategy suggests that it could be valuable for problem solving across a variety of mathematical domains, but the heuristic itself has not been specifically targeted as an area of study.

Little has been investigated about the use of smaller problems in the domain of counting problems. Eizenberg and Zaslavsky (2004) allude to such a strategy in their work on undergraduate students' verification strategies on combinatorial tasks. One of those strategies they identified was "Verification by modeling some components of the solution" (p. 26), and one aspect of such verification involved applying "the same solution method by using smaller numbers" (p. 26). Eizenberg and Zaslavsky provide an example of an expert mathematician effectively using this strategy, but they note that while the strategy "could be very powerful…it requires deep structural consideration." They go on to say that "We speculate that although it may seem natural to students to employ this strategy (as indeed some tried to), applying it correctly needs direct and systematic learning" (p. 32). However, researchers have not yet tied the particular strategy of solving smaller, similar problems to meaningfully developing students' facility with sets of outcomes.

Additionally, the paper is framed within Lockwood's model of students' combinatorial thinking (Lockwood, 2012), which explores the relationships between formulas/expressions, counting processes, and sets of outcomes. Lockwood notes that different counting processes impose different structures on the set of outcomes and suggests that meaningful progress can be made for students as they gain facility with the set of outcomes. This perspective, emphasizing sets of outcomes, underlies the work presented in this paper.

Methods

Twenty-two post-secondary students participated in individual, videotaped, 60-90 minute semi-structured interviews in which they solved combinatorial tasks. All of them had taken at least one discrete mathematics course, and some had taken courses in combinatorics or graph theory. Semi-structured interviews typically involve "an interview guide as opposed to a fully scripted questionnaire" (Willis, 2005, p. 20), and this methodology allowed for flexibility that could allow the interviewer to adapt to students' responses. The structure of the interviews was first to give students five combinatorial problems to solve on their own, during which time they were encouraged to think aloud as they worked. After they had completed work on the five problems, the students subsequently returned to a subset of these problems, and they were presented with alternative answers to evaluate. The motivation for this design was based on a desire to put students in a situation in which they had to evaluate incorrect but seemingly reasonable answers. Further details of the study can be found in Lockwood (2011a).

Tasks

While students in the study were given five tasks, due to space limitations only one problem, the "Groups of Students Problem," is presented. Below, both a correct answer and an incorrect answer are provided to facilitate subsequent discussion.

<u>The Groups of Students Problem:</u> *In how many ways can you split a class of 20 into 4 groups of 5?*

A correct answer to this problem is

$$\frac{\binom{20}{5}\cdot\binom{15}{5}\cdot\binom{10}{5}\cdot\binom{5}{5}}{4!}.$$

To arrive at this solution, five students can be chosen to be in a group, done in $\binom{20}{5}$ ways, then five of the remaining students are chosen to be in another group, $\binom{15}{5}$, then five more to be in a group, $\binom{10}{5}$, and then finally the last five to be in a group, $\binom{5}{5}$. However, the product must be divided by 4 factorial because the groups are not meant to be labeled or distinguished in any way – there is not a Group 1, Group 2, Group 3, and Group 4. Division by 4 factorial ensures that each solution gets counted once, as it should be. A typical incorrect solution neglects the division by 4 factorial.

Initial data analysis involved transcription of the videotape excerpts. Then, in line with grounded theory (Strauss & Corbin, 1998), the data was carefully coded for phenomena that could be organized into themes. What emerged were relevant instances of student work that highlighted students' uses of smaller, simpler problems, particularly as these related to students' work with sets of outcomes.

Results

The use of smaller, simpler problems arose a total of 15 times. Ten (six graduate and four undergraduate students) of the 22 students drew upon the strategy. While the strategy seems to have been underutilized among the students in the study, the strategy overwhelmingly helped the students who chose to implement it. Given students' overall difficulties on the problems, then, and given the fact that the use of smaller cases seemed to help some students, the strategy is worth examining as a potentially powerful aspect of combinatorial problem solving. The major result of this paper is that the use of smaller, simpler problems gave students greater access to sets of outcomes. That is, students were able to more clearly identify and manipulate outcomes when parameters were reduced, and this gave them leverage on the problems. For the sake of space, just two examples from the data are given here, both of which emphasize how the smaller case facilitated systematic listing of outcomes, which ultimately shed light on the original problem.

The first student, Mia, had initially arrived at an incorrect expression; she did not include division by 4 factorial. Upon revisiting the problem, she was asked to evaluate and compare her answer with the correct expression. Mia was thus in a position of comparing two different expressions to determine which was correct. Mia had some initial intuition about the role of 4 factorial, but she decided to attempt a smaller case in order to be sure. She worked through a smaller case of dividing six people into two groups of three. She first wrote down A, B, C, D, E, F to represent the people, and she wrote two circles with three dashes each in them. She noted that if she applied her initial method to the smaller case, she would get $\binom{6}{3}\cdot\binom{3}{3}=20$, and she stated that 20 "would not be too bad to write out." Mia then stated that if she applied the other expression to the situation, she would get 10. Next, Mia proceeded to write some examples out to see if she would get 20 or 10 as her answer for the small case. This is an instance in which Mia, in the context of the smaller situation, computed totals according to both possible expressions and compared the two. The smaller numbers allowed her to begin to write particular examples of outcomes, whereas within the original context this could not have been done feasibly.

Mia then wrote out divisions of six students into groups of two, and she wrote out ABC DEF, then CEF ABD, and then CDF ABE as possible divisions of the students. She paused and then wrote DEF ABC, and something significant happened: Mia noted that this was the same outcome as something she had already written – that is, ABC DEF was the same as DEF ABC. It seems that the smaller case (and specifically the smaller numbers) enabled Mia to write out some particular outcomes that she otherwise would not have been able to do (she had not written out such outcomes in her work on the original problem).

M: Um, alright A, B, C and then that forces DEF here. So that's one. ABD CEF. ABE and CDF. Hmm. <u>Let's see, oh right, because the first 3 could have been DEF, and then I would have been forced to put ABC in this group, but that's really the same, so these [referring to ABC DEF and DEF ABC] are really the same</u>...<u>Okay, yeah, I think that this double counts because if I just choose 3 people, it could have been A, B, and C, and then that forces DE and F in the second group. But, let's say the first three people were DEF, that forces ABC in the second group, and that's exactly the same, just, it doesn't matter, there's, there's, ABC are in a group and DEF are in a group.</u>

Mia explained the overcounting by referring to her initial solution; she identified two outcomes as being "exactly the same," as her language underlined above indicates. Mia was ultimately able to understand why division by 4 factorial in the original problem made sense and to identify the correct solution.

Our next student, Anderson, spent considerable time and energy listing out particular outcomes in the context of smaller cases. To scale back the original problem, Anderson decreased the number of groups and the number of options to make the problem more tractable. He started with dividing a class of four into two groups, and through listing found that there were three ways to do this. He then increased the problem to a class of six being split into two groups, and through careful systematic listing he found that there were 10 such possibilities. Anderson continued in this way, and at the heart of this work was pattern recognition – he was searching for a pattern in the numbers in order to generate the correct answer. While he ultimately ran short of time in the interview before Anderson could entirely finish this problem, he was on a very productive path toward making meaningful progress on the problem. When Anderson revisited the problem and was presented with the two common answers, he related them to the work and patterns he had generated initially. When it came to making sense of the solutions, then, it seemed as though his involved work of systematic listing and looking for patterns was instrumental in helping Anderson understand the problem generally.

Discussion

In prior work, Lockwood (2011a; 2012) indicated the importance of considering sets of outcomes for students as they count, suggesting that much benefit could be afforded by explicitly utilizing sets of outcomes in the activity of counting (see also Hadar & Hadass, 1981; Polaki, 2005). The findings in this paper build upon this notion, indicating that students' uses of smaller cases enabled them to engage with sets of outcomes through systematic listing.

In some instances, the use of a smaller problem allowed for work with the set of outcomes that might not otherwise have been attainable. The nature of counting problems makes them particularly appropriate for the strategy of using smaller, similar problems. Specifically, in counting problems, sets of outcomes are often so large that they can be difficult to conceive of and manage (for example, the answer to the Groups of Students problem is approximately 488 million). Smaller cases can reduce the magnitude of such sets and can make the problems and the solution sets more accessible.

Additionally, it is very important for a student to be able to articulate what he or she is trying to count, and smaller cases can facilitate such activity. In some counting problems the objects being counted can be quite difficult to articulate (in the Groups of Students problem, an outcome is one partition of 20 students into four groups of five; this necessitates coordinating a number of factors – 20 distinct students, what such a division might look like, and how they might be divided to create a desirable division). Smaller cases are particularly useful because they allow not only for the magnitude of the outcomes to be reduced, but often also for the outcomes themselves to be easier to identify. However, students must be aware of the fact that reducing a problem can introduce unexpected mathematical properties, and care must be taken when manipulating the original problem.

Conclusions and Avenues for Further Research

The results discussed above highlight the fact that solving smaller, simpler problems can allow students to work with sets of outcomes in meaningful ways. Overwhelmingly, the strategy helped the students who chose to implement it. Given student difficulties with counting problems, the use of smaller cases seems to be a promising domain-specific strategy that could be useful for students. One potential avenue for further research is to relate students' uses of smaller problems with their notions of what determines similarity among problems. That is, by identifying smaller, simpler problems, students can be thought of generating particular examples or instances of a given problem type. With increased attention on the role of examples in mathematics education literature (e.g., Bills & Watson, 2008), such an investigation could shed light on how students view examples of particular problem types. Another avenue to pursue is how to effectively generate this problem solving heuristic both among students and among pre-service and in-service teachers. This must be done with care, though, and students and teachers must be made aware of mathematical complications that can arise when reducing problems.

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