USING METAPHORS TO SUPPORT STUDENTS' ABILITY TO REASON ABOUT LOGIC

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In this paper, we describe an inquiry-oriented method of using metaphors to support students' development of conventional logical reasoning in advanced mathematics. Our model of instruction was developed to describe commonalities observed in the practice of two inquiry-oriented real analysis instructors. We present the model via a general thought experiment and one representative case study of a students' metaphorical reasoning. Part of the success of the instructional method relates to its ability to help students reason about, assess, and communicate about the logical structure of mathematical activity. In the case presented, this entailed a students' shift from using properties to describe examples to using examples to relate various properties. The metaphor thus imbued key example sequences with meta-theoretical significance. We introduce the term "wedge" to describe such examples that distinguish oft-conflated properties. We also present our analytical criteria for empirically verifying the specific influence of the metaphorical aspect of instruction.

Key words: Reasoning about logic, metaphor, real analysis, example use

Introduction

Philosophers classify logic as a form of meta-language because it describes the structure of more general forms of linguistic expression. It seems consistent then that reasoning specifically about logic is a form of metacognition. While the logical form of peoples' reasoning has been studied extensively, few studies carefully distinguish whether research subjects engage in linguistic analysis or metacognition, what we call *reasoning about logic*. Formalized logic, by definition, ignores the semantic content of statements or is at least generalizable beyond particular semantic content. Careful analysis of many previous studies on logical reasoning reveals that research subjects often reason about the semantic contents of statements or tasks, meaning the researcher may only observe the *logic of reasoning*. Since mathematics education studies show that students struggle with key logical structures in advanced mathematics like quantification (Epp, 2003, 2009; Roh & Lee, 2011), it seems important that students learn both standardized logical conventions and how to notice and assess the logical validity of their own mathematical reasoning. In other words, advanced mathematics students must develop tools for *reasoning about logic* and for transferring generalizable logical tools (ignorant of particular context) into their current context of reasoning (like sequence properties). This study addresses the following research questions:

1. How can mathematics instruction foster students' ability to reason about logical structure in mathematical activity and their adherence to logically valid forms?
2. How do metaphors for logical structure influence students' reasoning about particular examples in the context of real analysis?

The first question relates to our broader ongoing investigation of logic instruction in advanced mathematical courses. We present our findings from observations of two inquiry-oriented real analysis faculty via a thought experiment that models the common instructional method we observed in their classrooms. This method employed metaphors or stories to represent logical structure such that students could examine, assess, and communicate about such structure itself (*reason about logic*). The second question relates to the particular case we feature in this paper in which the professor used a metaphor to model and discuss the roles two key examples played in the class' theory of sequence properties.

Theoretical Backdrop

Our theoretical backdrop is inherently constructivist, more specifically aligned with the perspective called "radical" constructivism (von Glasersfeld, 1995). Learning should be characterized as students organizing their own mental and physical activity into schemes that support goal-oriented activity. The "transfer" or "application" of knowledge should not be viewed as use of an abstract tool for a specific situation, but rather treating a new situation as the "same" as a previously encountered one such that it is assimilated into the prior scheme of activity (ibid, 1995). Our characterization of "transfer" is consonant with Lobato and Siebert's (2002) notion of actor-oriented transfer and with Wagner's (2006) notion of transfer-in-pieces. We frame our questions of logic learning in terms of transfer to address the apparent paradox of how students can both reason about mathematics (semantic content) and reason about logical structure (ignorant of content). It should be pointed out that what we refer to as metaphors is often described in psychological literature as analogies. The term "metaphor" (treating one thing as if it were another) is used because it more closely matches our understanding of transfer in terms of assimilation. Our paradigm for metaphorical reasoning however closely parallels Holyoak's (2005) characterization of analogical reasoning.

The case study featured in this paper concerns the use of examples in proof-oriented mathematics, which has been extensively studied. Such examples can be used to refute a conjecture as a counterexample (e.g., Zazkis & Chernoff, 2008), to verify a conjecture or to understand a proof (Harel, 2001; Inglis, Mejia-Ramos, & Simpson, 2007; Weber, 2008; Weber, Porter, & Housman, 2008), to gain understanding of definitions (e.g., Alcock & Weber, 2008), and to elaborate concept images (e.g., Vinner, 1991). Whereas the example-related literature mentioned here often focused on examples students or mathematicians generate, our emphasis in this paper is on how professors can use metaphorical reasoning to guide their students' use of examples for formal mathematical activity.

Students' Reasoning about Logic versus the Logic of Students' Reasoning

Constructivism casts learning as organization of experience and action into schema that regulate human activity, rather than the acquisition of direct knowledge of external systems and phenomena. Research on students' logical activity indicates that their untrained reasoning (Evans, 2005) is not always consonant with formalized logic, but grows more sophisticated over time (Inhelder & Piaget, 1958). Logic then must be understood as an organization of human reasoning. Research upon logic must be careful not to deem reasoning as "illogical" when it differs from the formalized system of logic agreed upon by the mathematical or philosophical communities. Noting other researchers inherently sought to make "thought the mirror of logic," Piaget (1950) suggested, "simply to reverse the terms and make logic the mirror of thought, which would restore to the latter its constructive independence" (p. 30).

Students might be taught formalized logical systems to help them organize and systematize their naïve reasoning. In line with current research on "transfer" (Lave, 1988; Lobato & Siebert, 2002; Wagner, 2006) however, exposure to abstract systems does not guarantee formalized reasoning, especially because formal logic ignores semantic content. As Wagner (ibid.) said, "Abstraction [is] a *consequence* of transfer and the growth of understanding—not the cause of it" (p. 66). Thus in the paper, we investigate how students' learning about the logic of their reasoning "transfers" to their ongoing mathematical activity.

Description of the Study and Data Analysis

We embarked upon joint analysis of two real analysis classes in order to identify shared and effective tools or methods of instruction and learning that emerged in these classrooms. Both classes were inquiry-oriented in the sense that definitions, theorems, and proofs were treated as something to be constructed from intuitive meanings rather than pre-existing knowledge to be presented and internalized. Students were expected to play an active role in learning the materials by raising conjectures, justifying their own arguments, and debating contrasting claims within their discourse communities. Comparison of the two courses revealed that neither course dedicated time to "teaching logic" in isolation, but rather used a more localized or integrated approach to fostering logic learning. In line with the inquiry-oriented approach, the way the professors addressed logic guided students to explicitly examine logical structure such that they could reflect and guide their own logical activity.

Our data analysis followed a "grounded" approach (Strauss & Corbin, 1998) not in the fullest sense of that coding scheme, but in the sense of (1) developing a localized theoretical account that characterized the instruction and (2) testing such accounts by a constant comparative methods. We intend this model to appropriately characterize the instructional approach across different mathematical topics, logical structures, metaphor types (Dawkins, 2009), and the differing structures of the two inquiry-oriented classrooms (Dawkins & Roh, 2011). From our observations of teaching and learning episodes across the two classrooms, we abstracted a general model of teaching logical structure through metaphor or story. While two professors' practice does not constitute a large sample size, the model was vetted against a larger number of instructional episodes throughout the two classes (and multiple semesters). Due to space limitations, we present the general model in terms of a "thought experiment" in the sense of Freudenthal (1973, 1991) and a case study of one students' metaphorical reasoning about the logical (or meta-theoretical) structure of real analysis content.

The nature of the instructional practice we want to study poses a great methodological challenge because it is localized in the sense that logic was not taught as a topic, but rather as auxiliary to real analysis topics. Thus while the professors employed these tools throughout the semester, it is hard to separate the influence of that aspect of instruction from the range of other ongoing instructional activities related to the same mathematical topics. To trace the pedagogical influence of the metaphors, we must observe students' spontaneous and clear use of the metaphors for their mathematical activity. To avoid merely anecdotal evidence despite the methodological challenges, we sought "critical events" (Maher & Martino, 1996) from the two classes that satisfied the following criteria:

- C1. The students must spontaneously engage in metaphorical reasoning about logical structure or make a spontaneous mathematical (re-)discovery via metaphorical reasoning.
- C2. The students must elaborate the metaphor so as to influence their perception of the mathematical situation beyond simply using metaphorical "language."
- C3. The students must show evidence that they are reasoning about logical or meta-theoretical structure.

We shall justify how our featured case study satisfies each of these analytical criteria.

Results

Thought Experiment

To avoid what Wagner (2006) called "transfer by abstraction," students must somehow be led to explicitly examine the logic of their own mathematical activity. We claim both of the professors we studied successfully supported students' ability to reason about logic by the following instructional method. The instructor provided students with a metaphor that embeds the logical and/or quantitative structure within a quasi-real world context. The class explored the metaphor in conjunction with mathematical activity to induce a metaphorical mapping between the elements within the mathematical and metaphorical contexts. The context or story must be carefully created to display and motivate the logical structures within some frame of reference other than linguistic logic (such as deontic or rule-based reasoning does in the Wason, 1968, card task as discussed in Evans, 2005). That is to say the metaphor must have resonance, meaning it allows deep elaboration of the connections between the two domains (Black, 1962, 1977 as discussed in Oehrtman 2003, 2009). This distinguishes such instruction from that which Dubinksy & Yiparaki (2000) criticized where a mathematical statement's logic is simply compared to the logic of an everyday statement (which their research shows to be unreliable as a source of appropriate logical structure).

The metaphor then establishes a logical schema into which novel mathematical tasks may be assimilated. On such tasks, students assimilate the interrelationships and structure of a task into the schema of the metaphorical context, matching the known conditions that maximize transfer between superficially dissimilar tasks (Kimball & Holyoak, 2000). Figure 1 displays this form of instruction's intended avenue for students' logical transfer. In line with Wagner (2006), transfer occurs when a student has "constructed a framework of knowledge that was sufficiently complex [rather than abstract] to permit her to structure the two situations similarly" (p. 64). This differs from teaching formalized logics because it assumes students will see logical structure *within* the mathematical context rather than *abstracted from* it.

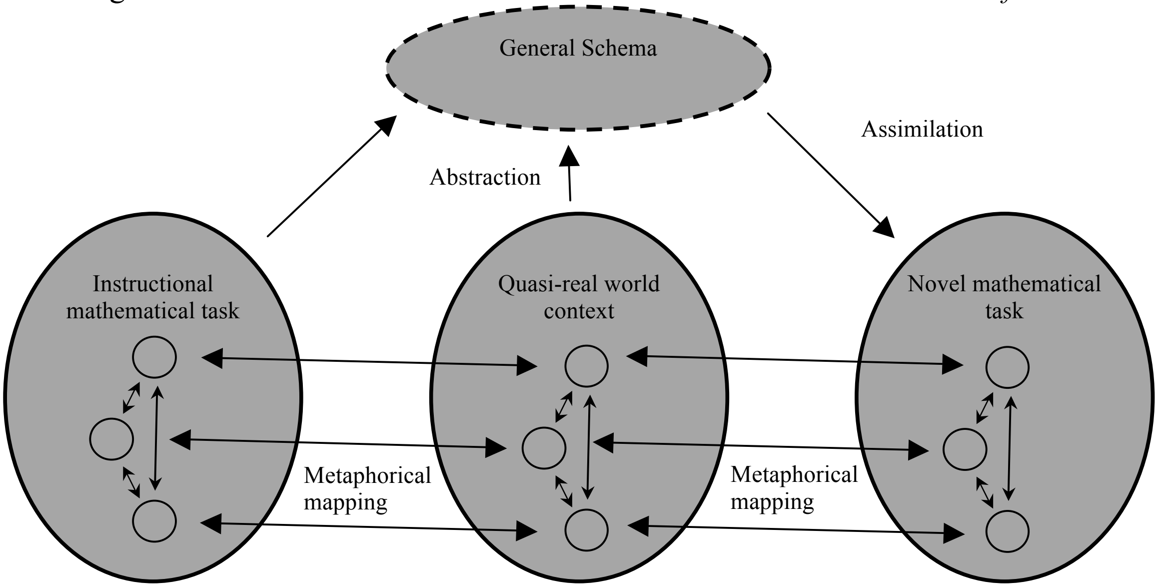

Fig 1. Pathway to transfer under metaphorical logical instruction.

Case study of Vincent's use of the Platypus metaphor

The real analysis class featured here was taught at a mid-sized, research university in the United States in the spring of 2008. A mathematician specializing in differential geometry taught the course. She had taught this real analysis at least 2 previous times and was awarded multiple teaching awards based upon her students' nominations. Most of the students were mathematics majors, a large portion of whom proceeded to pursue graduate degrees in mathematics or related sciences. The course met for 75 minutes twice per week over the course of a 15-week semester. The first author was present as an observer during all class meetings and conducted weekly task-based interviews with a small group of volunteers from each class. The researchers' interpretations of the professor's pedagogical intentions were vetted against her own articulations during bi-weekly interviews.

The professor used a large number of examples to guide students' reasoning about constructing definitions and theorems. She paid particular attention to the common student misconception that sequences only converge or tend to infinity if they are monotonic. To address this issue, she introduced the examples Penguin -{1,1,2,1,3,1,4,1,5,1…} and Platypus - {2,1,4,3,6,5,8,7…}. While she agreed that both appeared to tend to infinity, they both displayed strange properties. She specifically stated that biologists had to decide whether Platypus was a mammal because it did have hair, but it laid eggs instead of having live young. Ultimately, biologists agreed that platypus was a mammal just like the Platypus sequence does tend to infinity. Penguins look like they have fur, but they are birds and not mammals. Similarly the Penguin sequence is unbounded, but does not tend to infinity.

Once the class ratified a definition for a sequence tending to infinity, they verified that Platypus satisfied the definition and Penguin did not. The professor then provided the following true/false questions for the students to consider in groups and then discuss as a class (note that if a sequence tends to infinity or negative infinity, then it "properly diverges"): "If $\lim_{n\to\infty} x_n = +\infty$, then $\{x_n\}$ is unbounded and increasing." and "A sequence properly diverges if and only if it is unbounded." The students pointed out that the first statement is false because of platypus and the second statement "only works in one direction" because of penguin. Members of the class used the name platypus to refer both to the sequence itself and to its role as an atypical example of a sequence that tended to infinity. In contrast to *prototypes,* which are key examples that display the "standard" properties of a class of objects, we use the term *wedge* to denote examples that distinguish easily conflated properties. The wedge examples described in this paper are also different from the notion of boundary examples (Watson & Mason, 2001), which are examples used to make clear why a condition is required to define a concept by showing the definition would fail to describe the concept without the condition. While boundary examples relate a property and the category it describes, wedges are intended to relate multiple properties or definitions to each other.

For instance, the absolute value function is a standard wedge between the properties of continuity and differentiability. Platypus became the class' wedge between monotonicity and tending to infinity. Penguin acted as a wedge between unboundedness and tending to infinity. Students consistently referred to the sequences by their animal names without having to explain indicating the names became taken-as-shared.

An interview with Vincent two weeks later revealed he held a (nonstandard) personal concept definition (PCD) of proper divergence: "For every $K \in \mathbb{N}$… pick some term x_K right here, then for every term… $n > K$, $x_n > x_K$. So all the x_n's got to be in that interval [(x_K, ∞)] here." This definition would imply that sequences tending to infinity are monotone. However, when Vincent tried to describe Platypus, he recalled because of the name that it was a strange example of a sequence that tended to infinity. He thus modified his PCD to say you can only use even values of *K*. He also stated directly that he had previously thought sequences tending to infinity should be monotone, but that Platypus rendered that false.

When the interviewer asked Vincent one month later about various example sequences, Penguin was presented first. Vincent began listing properties of the sequence before citing its metaphorical name. When presented with Platypus, he immediately named the sequence explaining it was a "weird looking mammal." With some work, he elaborated the metaphor to say that mammals were sequences tending to infinity, but Platypus did so in a weird way. With some hesitance Vincent acknowledged that Platypus meant that not every sequence tending to infinity was monotonic. He restated his idiosyncratic PCD and noted again that Platypus limits the indices that "will work." It appeared that he "rediscovered" the distinction between the two properties via Platypus acting as a wedge.

Discussion

The Platypus and Penguin metaphors allowed the professor to simultaneously refer to the sequences themselves and their logical role as wedges between oft-conflated properties. The metaphors were localized tools for drawing attention to the logic of sequences categorization. The *logical metaphor* (Dawkins, 2009) induced a biological *structural metaphor* comparing the classification of sets to the classification of animals in biology. Vincent's repeated expression of the targeted misconception validated the professor's instructional intervention.

We argue that this episode satisfies all of our analytical criteria (denoted above by C1-C3) for evidence that the metaphor directly contributed to Vincent's mathematical learning. During the second interview, Vincent spontaneously recalled the sequence names and appeared to discover anew that sequences tending to infinity need not be monotone (C1). Without the metaphor, Vincent might likely have described Platypus as not tending to infinity according to his PCD. However, the name seemed to fix in his mind the fact that Platypus was a non-standard example of tending to infinity. Because he reevaluated his mathematical understanding, we claim that Vincent elaborated the metaphor beyond simple language use (C2). Third, though Vincent began the discussion in the latter interview *describing* sequences, he shifted to *categorizing* classes of sequences before actually *relating* sequence properties to one another. While previous research (Alcock & Simpson, 2002; Edwards & Ward, 2008) indicates that many students use definitions to *describe* examples, Vincent shifted to using examples to *relate* properties. This constitutes a sophisticated shift in the nature of his mathematical activity. Because he shifted from examining Platypus to the logical activity of relating properties, we claim that he was directly reasoning about the logic of sequence classification (C3).

This final claim also reveals how the instruction in this episode exemplifies the model described in our thought experiment. The metaphor guided Vincent to shift his attention to logical structure while still examining the particular properties of the mathematical objects at hand. His attention to the logical role of Platypus as a wedge allowed him to examine and assess his mathematical reasoning about particular sequence properties and his understanding of their definitions. Rather than abstracting his attention away from particular examples to talk about sequence properties, the metaphor elevated Platypus from a (non-)example of certain classes to a *wedge* with meta-theoretical significance. Figure 2 portrays Vincent's shift from seeing properties describing sequences to sequences relating various properties.

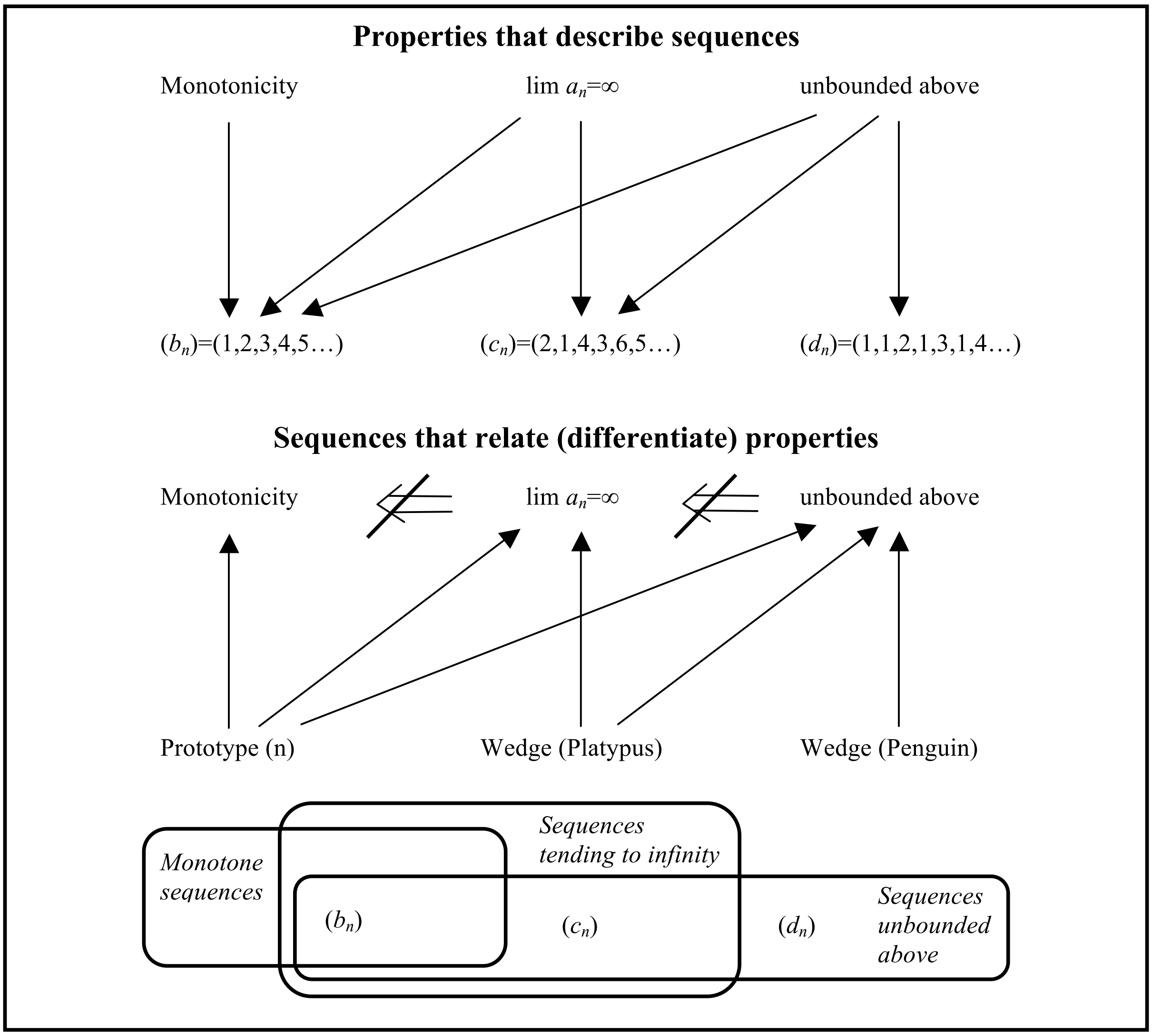

Fig 2. How metaphors shifted Vincent's reasoning about example sequences.

The form of instruction described here provided a valuable tool in the inquiry-oriented real analysis classrooms we observed for bringing logical structure into the consensual domain. The metaphors allowed students to examine and assess the logic of mathematical arguments. The method is somewhat limited because the metaphors model particular logical structures (such as wedges) rather than comprehensive logical systems such as propositional logic. They were generally used to support students' reasoning about problematic logical structures. We think one main analytical contribution is to draw attention to the issue of whether students are reasoning about logic at all. Future studies on students' logical reasoning should carefully delineate whether they are observing the logic of students' reasoning or students' reasoning about logic.

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