VERIFYING TRIGONOMETRIC IDENTITIES: PROOF AND STUDENTS' PERCEPTIONS OF EQUALITY

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This preliminary study explores how students' perceptions of the equality of trigonometric expressions evolve during the process of verifying trigonometric identities (VTI). If students already view the purported equality as being true, VTI may not offer much in the way of learning experiences for students. Using a semiotic perspective to analyze student work, this study attempts to describe the evolution of students' perceptions of identities upon application of VTI, focusing on the components of the student's VTI process that contribute to the evolution. Initial analyses of interviews conducted while students proved identities indicate that students are not fully convinced that the identities are initially true. However, successful VTI, signaled, for example, by the use of an idiosyncratic equality construction, endows equality on not only the purported identity but on ancillary equality statements generated as part of the VTI process.

Key words: Trigonometric identities, Proof, Equal sign, Semiotics

Verifying trigonometric identities (VTI) involves many domains of mathematics, such as the use of algebraic skills to manipulate expressions and the equality concept embedded in verification and identities. Additionally, while verification is proof-making, students engage in problem solving in order to complete the task. Yet, trigonometry, especially VTI, has been under-studied. Some studies focused on the medium through which students interacted with trigonometry while learning (e.g., Choi-Koh, 2003; Weber, 2005a). Other research described the understanding of trigonometric concepts and objects developed by students (e.g., Brown, 2005; Moore, 2010).

Few studies have explored the verification of trigonometric identities by students. For example, Delice (2002) investigated students simplifying trigonometric expressions. He found that students followed certain patterns of simplifying. However, the study assumed that students constructed no new knowledge during the simplification task, and thus students experienced no changes in their perceptions of the mathematical objects with which they operated.

Background

Historically, VTI has been a part of the trigonometry curriculum. However, the National Council of Teachers of Mathematics (NCTM) recommended scaling back trigonometric identity verification in the school-level curriculum (NCTM, 1989). The deemphasizing of VTI seemingly ignored NCTM's own statement that VTI "improves [students'] understanding of trigonometric properties and provides a setting for deductive proof" (NCTM, 1989, p. 165). In a critique of the trigonometry standard and the standards overall, Wu (n.d.) wondered how average, non-college bound students were to understand identities such as $\sec^2 x = 1 + \tan^2 x$ if not given the opportunity to prove them. Additionally, the more recent Common Core State Standards for Mathematics do not explicitly suggest the verification of general identities; strands F-TF 8 and 9 address proving the Pythagorean identity and the addition and subtraction formulas and then subsequently using them to solve problems (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

VTI is an individualized process that a person utilizes to demonstrate, to himself and to an external audience, that one trigonometric expression is equivalent to another trigonometric expression. Harel and Sowder (1998) defined *proving* as a "process employed by an individual to remove or create doubts about the truth of an observation" (p. 241). Thus, VTI is a proving process. Within VTI, the theorem to be proved is the purported equality of the two trigonometric expressions. Henceforth, this purported equality will be referred to as the *theorem identity*. The theorems used by a person to prove the theorem identity are equivalent expressions or identities; hence, the theorems of VTI are statements of equality.

Viewing VTI as proof construction suggests exploration of student notions of the theorem identity and equality. As noted by Weber (2001), studies have shown that inadequate understanding of the mathematical content of a theorem affects correct use of the theorem. Additionally, as the theorems in VTI are equality statements, student understanding of the theorems includes understanding of equality and the *equals* symbol. Studies have found that students of all ages, even within the college ranks, may view the equal symbol as operational rather than relational (Kieran, 1981; Weinberg, 2010). Furthermore, weak notions of the equal symbol may affect performance in algebra (Alibali, Knuth, McNeil, & Stephens, 2006). These weaknesses may in turn affect a student's ability to use the identities during identity verification.

Finally, CadwalladerOlsker (2011), quoting de Villiers, stated that one role proof serves is that of the verification of a theorem. Thus, when a student verifies the theorem identity to be true, he or she has proven that the purported equality is indeed true, and now that equality may be treated as a theorem. However, "novice proof writers often complain that it is pointless to prove theorems that 'everybody knows,' or that have already been proven in the past" (CadwalladerOlsker, 2011, p. 40). If students engaged in VTI have this attitude of the theorem identity and of VTI, then the question exists of whether or not students can make conceptual gains through VTI. If students view the theorem identity as already true, does VTI really offer a learning experience for them? If no need exists for students to verify a truth, will they experience a change in their perceptions of the objects with which they work?

Purpose

Tall (2002) suggested that the process of proving a statement encapsulates the theorem as a concept. Once encapsulated, the mathematical object may be used to build new knowledge and prove more theorems. This encapsulation process may occur in VTI as students verify the equality statements, creating new identities. In other words, students should experience a shift in their perceptions of the theorem identities and enhance their understanding of identities.

The purpose of this present study is to describe the evolution of students' perceptions of the theorem identity upon application of VTI, focusing on the components of the student's VTI process that contribute to the evolution.

Theoretical Framework

As suggested by Weinberg (2010) in his study of students' conceptions of equality, in this present study, a semiotic perspective is used to analyze students' VTI solutions. Semiotics places the emphasis on the connections between the representation of ideas and how these representations influence the activity of solving the problem. In other words, as a student writes his VTI solution, how he writes the solution is intertwined with his notions at play in the process of VTI; the representations and the notions affect each other.

Methodology

This study uses a pragmatic approach to research, placing primary importance on the research questions and using methods of data collection and analysis which best address the question (Creswell & Plano Clark, 2011). For this reason, qualitative methods were used to explore students' conceptions as they solve problems.

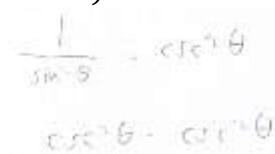
During the spring 2012 semester, 33 students enrolled in a trigonometry course at a large, Midwestern university participated in a study of their problem solving behavior while verifying trigonometric identities. About two weeks after the conclusion of the unit on VTI, and after a semester exam that included VTI items, 8 students agreed to be interviewed as they verified trigonometric identities. While solving the problems, the students were encouraged to think aloud, providing their motivations for their decisions and actions. During the interviews, students' conceptions of identities, equality, and VTI were explored through follow-up questions. The interviews were recorded using a digital voice recorder and were subsequently transcribed. The transcripts of the interviews were analyzed using a grounded, open-coding approach (Strauss & Corbin, 1998).

Preliminary Results

Certain features of VTI appeared to transform students' perceptions regarding the equality of trigonometric expressions. Prior to engagement in VTI, some students were skeptical of the equality purported by the theorem identity. According to one student, Amber, "There's a lot of skepticism in the beginning. … If I don't see those steps in my head, or if they don't look the same in the beginning, then I'm basically trying to prove it wrong." Other students viewed the expressions as equal, but the equality was attenuated by viewing it as tentative or assumed; they did not hold strong convictions concerning the equality. Another student, Bella, stated, "I haven't shown that in my own head that they're equal. I'm trusting that they're equal. … It's just I'm assuming they're equal until I say, until I have shown that they're not. But you don't know for absolute positively sureness that they are equal." Upon application of VTI, skepticism, doubt, or tentativeness was removed for the students, and the students believed they could unequivocally assert that the expressions were equal.

The transformation of the theorem identity's status occurred in both a public and private fashion and depended upon how the students structured their verifications. Some students mentioned accepting the truthfulness of the theorem identity due to an unraveling process in their minds. For others, the shifting of their perceptions occurred when they wrote a reflexive statement, such as $B = B$, for their final VTI step. As Amber put it, "In my mind, it's verified, once they look, I mean, once they are the exact same on either side, then for me, it's verified." This *reflexive step* was an important and necessary step for some students to successfully conclude VTI. It signaled the beginning of the transformation, although for some students, the reflexive step seemed somewhat ritualistic. Nevertheless, through these experiences, students ascertained for themselves the truth of the theorem identity.

For their written solutions, most students employed a columns method, manipulating one expression and cascading the manipulated expressions down to form a column. A few students rewrote the unchanged target expression and connected this expression with an equal sign to each of the manipulated expressions in the column, creating a series of equality statements. While seemingly stating that the expressions were equal, students tended to believe the expressions were not equal until VTI was completed, for example, with the writing of the reflexive step. At this point, in addition to the theorem identity, all of the intermediate equalities that had been written could be retroactively declared as true. For example, consider the work of Alan in verifying the identity $1/(1-\cos^2\theta)=\csc^2\theta$:



In describing his use of the reflexive step, Alan commented, "To me that's just verifying that I was able to get the correct answer and to show whoever is looking at this that if you eventually get to the steps, you will get what equal the cosecant squared equals cosecant squared. It's pretty much to me just to show that I'm done, … to kind of put any doubt out of peoples' minds." In referring to the initial "equality" written, Alan clarifies, "Well, that is true because I was eventually able to work it down. … I got, was able to get cosecant squared on the other side, that, they ended up being true." Thus, reaching the reflexive step not only indicated that the theorem identity was true; Alan was able to consider the first "equality" written a true equality as well.

Audience Questions

1. During VTI, equality appears to have a time-dependent nature. What theories or frameworks exist to describe this behavior?
2. Although this study views VTI as proof-making, doing so seems somewhat naïve. What are some other frameworks that could adequately capture student behavior in VTI?

References

Alibali, M. W., Knuth, E. J., McNeil, N. M., & Stephens, A. C. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for research in mathematics education*, *37*(4), 297-312.

Brown, S. A. (2005). The trigonometric connection: Students' understanding of sine and cosine. (Unpublished doctoral dissertation). Illinois State University, USA.

CadwalladerOlsker, T. (2011). What do we mean by mathematical proof? *Journal of Humanistic Mathematics*, *1*(1), 33-60.

Choi-Koh, S. S. (2003). Effect of a graphing calculator on a 10th-grade student's study of trigonometry. *The Journal of Educational Research,* 96(6), 359-369.

Creswell, J. W., & Plano Clark, V. L. (2011). *Designing and conducting mixed methods Research* (2nd ed.). Thousand Oaks, CA: SAGE Publications.

Delice, A. (2002). A model of students' simplification of trigonometric expressions. In S. Goodchild (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *22*(1&2), 19-24.

Harel, G. & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In E. Dubinsky, A. Schoenfeld and J. Kaput (Eds.), *Research in Collegiate Mathematics Education, III* (pp. 234-283). Providence, RI: American Mathematical Society.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326.

Moore, K. C. (2010). The role of quantitative and covariational reasoning in developing precalculus students' images of angle measure and central concepts of trigonometry. In *Proceedings of the 13th Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. National Governors Association Center for Best Practices, Council of Chief State School Officers: Washington D.C.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research. Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage Publications.

Tall, D. (2002). Differing modes of proof and belief in mathematics. In F.-L. Lin (Ed.), *Proceedings of the International Conference on Mathematics,* Taipei, Taiwan, National Taiwan Normal University, pp. 91-107.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101-119.

Weber, K. (2005a). Students' understanding of trigonometric functions. *Mathematics Education Research Journal, 17*(3), 91-112.

Weinberg, A. (2010). Undergraduate students' interpretations of the equals sign. In *Proceedings of the 13th Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC.

Wu, H. (n.d.). The mathematics education reform: What is it and why should you care? Retrieved from http://www.math.berkeley.edu/~wu/reform3.pdf