STUDENTS' EMERGING UNDERSTANDINGS OF THE POLAR COORDINATE SYSTEM

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The Polar Coordinate System (PCS) arises in a multitude of contexts in undergraduate mathematics. Yet, there is a limited body of research investigating students' understandings of the PCS. In this report, we discuss findings from a teaching experiment concerned with exploring four pre-service teachers' developing understandings of the PCS. We illustrate ways students' meanings for angle measure influenced their construction of the PCS. Specifically, students with a stronger understanding of radian angle measure more fluently constructed the PCS than their counterparts. Also, we found that various aspects of the students' understandings of the Cartesian coordinate system (CCS) became problematic as they transitioned to the PCS. For instance, mathematical differences between the polar pole and Cartesian origin presented the students difficulties. Collectively, our findings highlight important understandings that can support or prevent students from developing a robust conception of the PCS.

Key Words: Polar Coordinate System, Cartesian Coordinate System, Pre-service Secondary Teachers, Teaching Experiment, Quantitative Reasoning

Introduction

In addition to having many mathematical applications, polar coordinates have a multitude of real-world applications, some found in physics and engineering (Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007). Often, students are introduced to polar coordinates in pre-calculus, where the emphasis tends to be on geometric interpretations and conversions between Cartesian and polar coordinates (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009). Polar coordinates later arise in single-variable calculus where students are asked to determine areas contained by polar curves using integration and again in multi-variable calculus where they are used to model three-dimensional objects using spherical and cylindrical coordinates.

Available research, which is sparse, on students' understandings of polar coordinates mainly focuses on student conceptions and misconceptions about the *polar coordinate system* (*PCS*). However, there is little research investigating problems that arise as students *construct* the PCS, which has inherently different features than the *Cartesian coordinate system* (*CCS*). In order to better understand students' conceptions of the PCS, we explored four secondary pre-service teachers' (who we refer to as students) thinking using a teaching experiment methodology (Steffe & Thompson, 2000). We used this methodology to investigate the questions: (a) What ways of thinking do students engage in when constructing the PCS? (b) What issues arise during students' construction of the PCS? (c) How do students utilize their understanding of the CCS while learning the PCS? and (d) How do conventions of the CCS influence students' understanding of the PCS?

In the present work, we focus on issues that arose when students were developing meanings of the PCS. For example, students' meanings for angle measure influenced their conception of the PCS. Further, consistent with previous research, we found some students relied on rules and conventions from the CCS when plotting points or interpreting graphs in the PCS. Against the backdrop of such issues, we highlight ways of reasoning that emerged as important for the PCS.

Background

Research has shown collegiate students often struggle with the PCS, both in mathematical and real-world application problems (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007). Sayre and Wittman (2007) identified that engineering students with a stronger understanding of the PCS used the PCS to simplify real-world problems while students with a weaker understanding of the PCS relied on the CCS, often to their own detriment, in similar situations. This research points to the importance of students developing a robust understanding of the PCS in order to interpret and solve problems that arise in real-world contexts, particularly in engineering and applied fields.

Montiel et al. (2008) investigated second semester calculus students' understanding of the PCS by focusing on students' function concept and how they adapted their definition within the PCS. They found many students equating the vertical line test with the definition of function, which became problematic in the PCS. For instance, students applied the vertical line test to the polar graph of $r = 2$ (forming a circle) to conclude the graph was not a function. Others first converted the equation to its CCS form before applying the vertical line test; again concluding the graph did not represent a function. Even students who reasoned that a function is a relation where each input is mapped to one output struggled when attempting to interpret relations in the PCS. Montiel et al. (2008) also noted that students struggled with the notational convention of the input appearing after the output in ordered pairs when asked to graph functions of the form $r(\theta)$. Summarizing their findings, they claimed, "some misconceptions about functions in rectangular coordinates transfer to polar coordinates, and that some new misconceptions related specifically to the polar coordinate system arise" (p. 62 Montiel et al., 2008).

The aforementioned study (Montiel et al., 2008) was a precursor to a larger study exploring multi-variable calculus students' conceptions of the three-dimensional rectangular, polar, and cylindrical space (Montiel, et al., 2009). Both studies (Montiel et al., 2008; Montiel et al., 2009) emphasized the importance of students developing a robust understanding of the various representations of functions in different systems, as well as the ways in which conventions played a role in students' mathematical understandings. Both relied on comparing their research to current calculus textbooks as a way to compare and contrast their findings with practices supported by these text, and both articles were mainly concerned with students' conception of function in interaction with multiple coordinate systems. We extend this work by looking at issues that arose when students construct the PCS. We conjectured that aspects of the PCS could be problematic because of the way in which the PCS is defined. For instance, whereas the CCS is based on two directional lengths, the PCS includes a coordinate that represents an angle measure, and thus how a student conceives the PCS will be reliant on their angle measure meanings.

Quantitative Reasoning and the Polar Coordinate System

In addition to leveraging available research on the PCS, we also rely on theories of *quantitative reasoning* (QR) (Smith III & Thompson, 2008; Thompson, 2011) to inform the study. QR takes the stance that quantities are cognitive constructions and therefore should not be taken as a given (Thompson, 2011). As an illustration of QR, consider the role of angle measure in the PCS. Previous research (Moore, 2012) has illustrated that QR plays an important role in students' meanings for angle measure, including supporting students in coming to understand angle measure as an equivalence class of arcs. Because the PCS entails a coordinate that conveys an angle measure, we conjectured that students' meanings for angle measure will contribute to their capacity to construct and apply the PCS. For instance, consider the set of points defined by $(r,2)$ in the PCS; the radial component defines a set of points r units from the pole and the angular component defines a set of points forming a ray at an angle of measure 2 radians counter-clockwise from the polar axis. When viewing the set of points defined by $(r,2), r>0$, with an understanding of angle measure rooted in arcs, one can conceive *every* arc subtended by the formed angle as having a length of 2 radii when measured in a magnitude equivalent to that circle's radius (Figure 1); the ray defines an equivalence class of arcs upon which the coordinate system is based. On the other hand, as we later illustrate, without understanding the ray as defining an equivalence class of arcs tied to measuring in radii, one is left with arcs of different lengths being used to represent what is supposed to be a constant value. This image can lead to interpreting points along a ray as having different angular (arc) values.

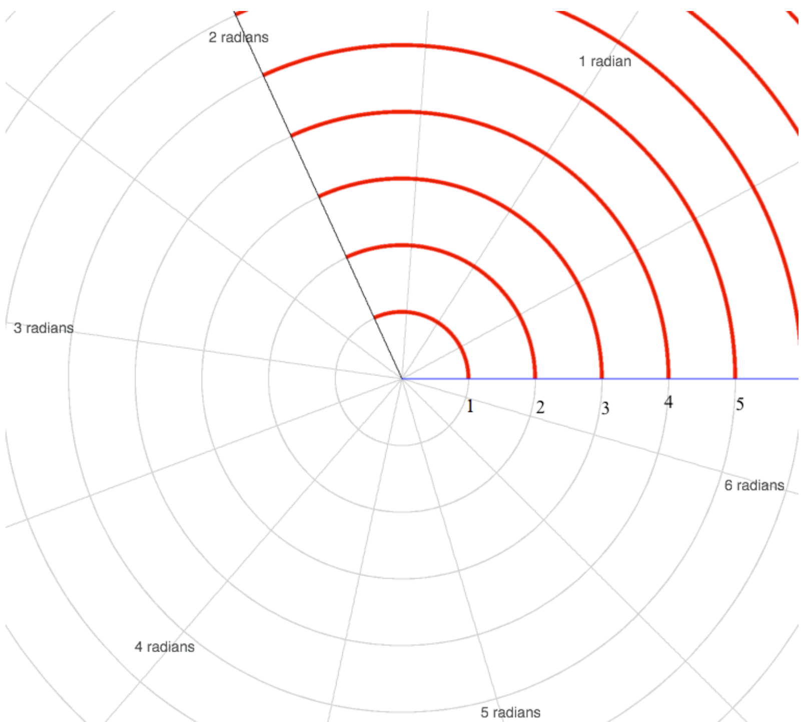

Figure 1. Polar coordinates and the angular component defining an equivalence class of arcs.

Methods and Subjects

The four participants (Katie, Jenna, John, and Steve) of the study were third-year (in credits taken) undergraduate students at a large state university in southeast United States, enrolled in a pre-service secondary mathematics education program. Prior to the study, all students had taken mathematics courses through at least Calculus II. We chose the four students from a pool who volunteered based on results from a pre-assessment given to students at the beginning of a content course and homework evaluations for the first month of this class. Throughout the study, the students worked in pairs (Steve & Jenna / Katie & John). Based on their homework evaluations and pre-assessment, we paired two higher performing students (Katie & John) and two lower performing students (Steve & Jenna). We conjectured pairing in this manner would help us identify different ways of thinking about the PCS.

Our intention in the study, based on a radical constructivist epistemology (Glasersfeld, 1995), was to build and test models of the students' thinking. To accomplish this goal we conducted a teaching experiment (Steffe & Thompson, 2000) with each pair of students working together with one teacher-researcher while an observer-researcher videotaped each of the sessions. Over the course of three weeks, the teaching experiment spanned five sessions, each approximately one hour and fifteen minutes long.

All of the videos were transcribed and then reviewed individually by the research team. When analyzing the data, we used a combination of an open and axial coding approach (Strauss & Corbin, 1998) and conceptual analysis (Thompson, 2000). We first analyzed each pair of students' words and actions in order to characterize their thinking. We further analyzed common and often unexpected themes which arose in our analyses. Upon our characterization of these themes, we searched the data for instances confirming or conflicting with our models of the students' thinking. We sought to build more viable models (Steffe & Thompson, 2000) of the students' thinking, including shifts in their thinking, through this iterative process.

Results

Through our conceptual analysis, many themes in the students' activity emerged. We focus our results around two themes. The first theme centers on the importance of students' radian angle measure meanings in the PCS. The second theme deals with characteristics of the polar and Cartesian systems, and how the students handled various conventions of the systems.

The Importance of Radian Angle Measure

The first activity in the teaching experiment asked the students to determine what information a surface radar system on a ship, with the ship defining a center point, might give in order to determine the exact location of another object at sea. We posed that the system provides how far a detected object is from the ship and asked the students to determine a second quantity that would define an exact location of the object. Our goal was for students to realize a distance from the fixed point and an angle measure (with a defined reference ray and rotation direction) suffices to provide an exact location. At the point in which the students determined these quantities, our goal was to task them with constructing a coordinate system based on these quantities (e.g., the PCS).

When comparing the two pairs of students' construction of the PCS over the course of the task, it became apparent that a robust understanding of angle measure was critical to their work on the task. Prior to the teaching experiment, students received instruction intended to develop a strong understanding of radian angle measure using curricula based in research on the topic (Moore, 2012). Unsurprisingly, the students' work on the radar task established that the students had developed different understandings during the angle measure lessons.

Both Katie and John exhibited an understanding of radian measure consistent with that described above (e.g., equivalence of arcs based in measuring in radii) when constructing the PCS. As such, Katie and John experienced little difficulty constructing a coordinate system that involved an angle measure and a radial distance, and often used measuring along arcs to give meaning to angle measure coordinates. Contrarily, early in the teaching experiment, Steve and Jenna experienced difficulty constructing the angular component in the PCS.

Beginning the task, Steve and Jenna raised the idea of using angle measure to provide a second quantity for the radar system. The pair chose a radial unit, which represented a fixed magnitude, and drew several concentric circles based on this unit. They experienced a perturbation when attempting to reason about points along a single ray as being an equivalent arc from the reference ray. With a fixed unit length in mind, Steve and Jenna interpreted arcs on different circles but subtended by the same angle as producing different coordinate measures (Figure 2). Thus, the students questioned using angle measure, as they had conceived, as an appropriate coordinate.

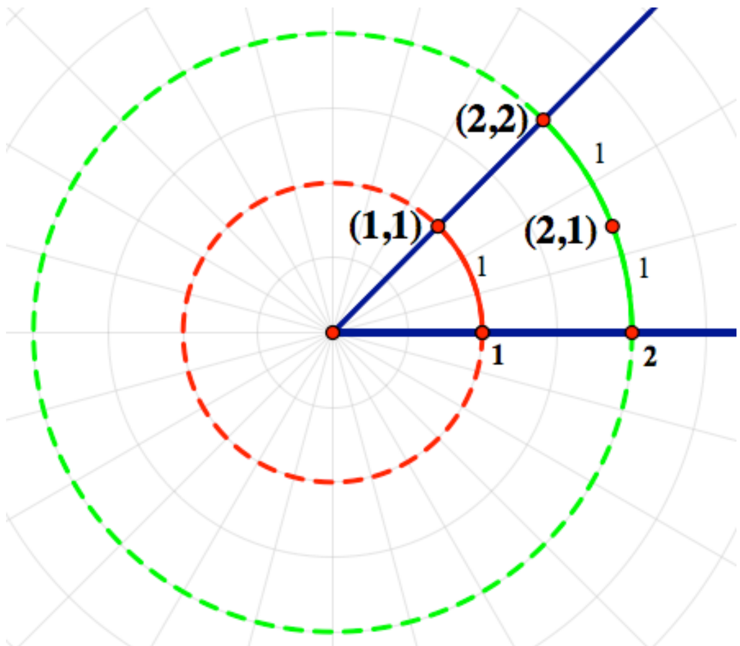

Figure 2. Using a fixed magnitude to measure arcs.

As the teaching experiment progressed, we worked with the students to develop radian angle measure as defining an equivalence class of arcs. As the pair constructed such an understanding of angle measure, they both made strides in their PCS concept. However, this progress was not trivial. For instance, Jenna continued to struggle coordinating angle measures and the measures of subtended arcs. At one point she stated, "See that's what confuses me, because how can we say that the length, I mean it's, I understand that [the arc length] is the same length as one radius, one radian. But the angle measure is also one radian, so they're all (*referring to several subtended arcs*) one radian?" After a discussion that entailed identifying that "one radian" refers to *all* of the arcs subtended by the angle, she explained that she had difficulty reasoning about an angle measure in terms of measuring arcs. Collectively, the students' difficulties with angle measure corroborate Moore's findings (2012) about students' images of angle measure, while highlighting the importance of students' meanings for angle measure when learning the PCS.

Polar and Cartesian Characteristics

We also noted the students encountering perturbations when considering outcomes from the PCS with the CCS, especially when trying to make a connection between the origin in the CCS and the pole in the PCS. For example, an issue arose when we gave them a graph of $r(\theta) = 2\theta - 0.5$ and asked them to determine a function rule to define the graph.

After finding the rate of change of the relation using points on the graph, Steve used the fact that the graph passed through the "origin" (e.g., pole) to determine the rule of $r(\theta) = 2\theta + 0$, with 0 representing that the graph passes through the "origin." However, when the students tested their rule with another point, they found that the constant is -0.5 rather than zero (e.g., $r(\theta) = 2\theta - 0.5$). This led the students to assume that the given graph was misleading and never actually passed through the "origin." Furthermore, when asked if the graph did or did not pass through the "origin," Steve replied, "It doesn't go through the origin. 'Cause that wouldn't match our function (*referring to the rule*)." When probed as to why it "wouldn't match our function," Steve argued, "If you use the origin zero-zero, then if your radius is zero and your theta is zero, there's no way you can get zero unless b is zero." This interaction suggests that Steve encountered a perturbation with his interpretation of the "origin" in the PCS as being the same as the origin in the CCS, namely the ordered pair (0,0). Steve, not realizing the pole in the PCS is represented by all pairs of the form $(0,\theta)$, believed passing through the "origin" in the PCS required that the function represent the pair (0,0). Thus obtaining a rule that did not entail this pair led him to conclude that the given graph could not pass through the pole as he had conceived it.

Later in the conversation and after the researcher followed up with a discussion of the points represented by the pole in the PCS, Steve explicitly stated, "It's so hard to grasp that it goes through the origin but it's not zero-zero. I feel like that's ingrained in our mind," further indicating Steve's experience with the CCS influenced his understandings of the PCS. Specifically, Steve's concept of the origin from the CCS was a problematic influence on his understanding of the pole in the PCS, providing an example of an issue that might arise when students apply conventions from the CCS with little consideration of the differences between the two systems that stem from their underlying structure.

Another issue that arose was that the students were irked by the mathematical convention of writing the coordinate point in the PCS as (r, θ), which they interpreted as (output, input), due to initially working only with functions of the form $r(\theta)$. To the students, this contradicted the CCS convention of writing pairs in the form (input, output). After adopting the (r, θ) convention, Jenna and Steve then exhibited discomfort when equations were given with θ as a function of *r*, instead of *r* as a function of θ. For instance, when asked to graph $\theta = r^2$, both chose values of θ and found corresponding values of *r* using the rule $r = \theta^2$. After some questioning, Steve caught their error: "It's theta equals r squared, so wouldn't r be the square root of theta, right? So for every theta you would get the square root of that would give you your radius." In spite of his awareness of their error, he desired to rewrite the equation so that θ formed the input of the function ($r(\theta) = \sqrt{\theta}$). After substituting in the point (4,2) to $\theta = r^2$ to convince Jenna of their error, Jenna claimed, "We just did the math backwards." Steve confirmed that the roles of the variables had switched in the given rule (e.g., the given rule implied that θ is a function of *r*), which conflicted with their prior experiences that defined (r, θ) as (output, input). Steve was later able to identify such conventional practices as being arbitrary decisions on the part of the practitioner, but his actions indicated that he remained fixated on a convention (e.g., (r, θ) as (output, input)) he adopted from early experiences with functions in the PCS.

Conclusions and Implications

Research suggests students are often given a superficial presentation of the PCS at the precalculus and single-variable calculus level (Montiel et al., 2008; Montiel et al., 2009). This approach to the PCS likely leaves students with impoverished PCS concepts, some of which stem from problematic connections to the CCS. Our results support previous findings concerning the influence the CCS has on student understandings of function in the PCS (Montiel et al., 2008; Montiel et al., 2009). While previous research identified students' (mis-)use of the vertical line test in the context of the PCS, our results indicate that other issues stem from certain features of the PCS. For instance, it was important that the students understand the pole of the PCS as represented by an infinite number of coordinate pairs in the PCS. In the case that the students conceived the pole as represented by a unique pair (e.g., (0,0), the CCS origin), they had difficulty reconciling the graph of a function in the PCS and the function rule. As another example, students' understandings of angle measure were critical to their PCS concept. In the CCS, directed lengths form both coordinate quantities. However, in the PCS, an angular component forms one of the quantities. It follows that ideas of angle measure, including equivalence of arcs, emerged as central to the students' progress.

Future researchers may be interested in exploring how to support students in overcoming various conventions of the CCS and PCS when attempting to make connections among these systems. Our findings, in combination with previous research (Montiel et al., 2008; Montiel et al., 2009), suggest that if students' experiences are predominantly within one coordinate system, then their understandings for related concepts (e.g., function) become inherently tied to that coordinate system. Future research that investigates using a multitude of coordinate systems in the teaching of mathematics might prove useful for determining critical ways of reasoning about function and how to promote these ways of reasoning.

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