PROVIDING OPPORTUNITIES FOR COLLEGE-LEVEL CALCULUS STUDENTS TO ENGAGE IN THEORETICAL THINKING

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Previous research has reported an absence of a theoretical thinking component in college-level Calculus courses. While valid arguments can be made for or against the necessity of incorporating such a component, our belief is that students who wish to engage in theoretical thinking should be given the chance to do so. Our goal is to determine whether instructors can provide their students with opportunities to engage in theoretical thinking, despite constraints they often face such as time, course content, and assessment material. This report presents a preliminary analysis of a study in which we presented students in a Calculus class with tasks intended to provoke theoretical thinking. Using Sierpinska et al.'s (2002) model for theoretical thinking we show that students who engaged in these optional tasks where in fact engaging in theoretical thinking. We conclude that, despite the institutional constraints, incorporating such a component is indeed feasible.

Keywords: Theoretical thinking, Calculus, Quizzes, Institutional constraints

This paper presents preliminary results of an ongoing research. The study was conducted in a prerequisite Calculus class (college-level) offered at a university in Montreal. These are multi-section courses taught by different instructors but designed by a single course examiner. The course examiner writes the outline of the course, the weekly assignments, and the midterm and final exams. In addition to specifying the topics to be taught each week, the outline includes a list of "recommended" exercises from a common-assigned textbook which are indicative of the types of problems that will appear on the weekly assignments and midterm and final exams. The problems on these assessments can be described as "routine" problems in the sense of (Lithner, 2003; and Selden et al., 1999). Therefore, instructors of these courses face several constraints: a fixed outline (i.e., they cannot change it), with a fixed order for delivering the content, a pre-chosen set of exercises, assessments that they cannot modify; plus constraints associated to classroom time and class-size. Our goal is to explore whether in a course and setting such as the one described here, and despite the mentioned constraints imposed on the instructor, students can be provided with opportunities to engage in theoretical thinking. To answer this question, we propose and investigate the effectiveness of one means to engage students in theoretical thinking, described further below.

The motivation behind this research is the reported absence of a theoretical thinking component in Calculus courses; in fact, previous research describes Calculus students' predominant behaviors that are not indicative of theoretical thinking (e.g., Boesen et al., 2010; Hardy, 2010; Lithner, 2000; Selden et al., 1999), reporting a link between students' behaviors and the tasks set before them. Moreover, the development of mathematical reasoning skills, personal sense-making, and convictions of mathematical concepts are seen to arise from interactions that take place in the classroom (e.g. Yackel, 2004; Hanna, 1991); this study addresses these results particularly.

The study was conducted over a thirteen-week semester and the tool that was used to foster theoretical thinking was *weekly quizzes*. The quizzes consisted of one or two questions that were designed in a way such that meaningfully answering a question would require students to think theoretically. To determine whether the quizzes were successful in engaging students in theoretical thinking, the model developed by Sierpinska et al. (2002) was used after modifying and adapting it for this study.

Theoretical perspective

With a concern for the individual and a desire to characterize theoretical thinking, Sierpinska et al. (2002) were inspired by Vygotsky's work; namely his distinction between scientific and everyday (or 'spontaneous') concepts (1987). In particular, Vygotsky characterized scientific concepts as formed in the mind on the basis of concretization from general statements, as opposed to spontaneous concepts that are formed on the basis of generalization and verbalization from concrete experience. Furthermore, Vygotsky argued that theoretical thinking does not develop spontaneously in children as one last "stage" of their cognitive development, but requires special nurturing. Based on these assumptions Sierpinska et al. (2002) constructed a model of theoretical thinking, shown below, in which the main postulated categories of theoretical thinking are "reflective", "systemic", and "analytic" thinking. The authors presented features of theoretical thinking that were relevant to their study in a Linear Algebra class and operationalized the model with theoretical behaviors which, when displayed by a subject, are indicative of the occurrence of theoretical thinking.

Category of TT **• Feature of TT**	**Description**
TT1 Reflective	Theoretical thinking is thinking for the sake of thinking.
TT2 Systemic	Theoretical thinking is thinking about systems of concepts, where the meaning of a concept is established based on its relations with other concepts and not with things or events.
• TT21 Definitional	The meanings of concepts are stabilized by means of definitions.
• TT22 Proving	Theoretical thinking is concerned with the internal coherence of conceptual systems.
• TT23 Hypothetical	Theoretical thinking is aware of the conditional character of its statements; it seeks to uncover implicit assumptions and study all logically conceivable cases.
TT3 Analytic	Theoretical thinking has an analytical approach to signs
• TT31 Linguistic sensitivity	
o TT311 Sensitivity to formal symbolic notations	
o TT312 Sensitivity to specialized terminology	
• TT32 Meta-linguistic sensitivity	
o TT321 Symbolic distance between sign and object	
o TT322 Sensitivity to the structure and logic of mathematical language	

Table 1 – Sieprinska et al.'s (2002) model for theoretical thinking

The learning of Calculus and of Linear Algebra likely prompts different aspects of theoretical thinking due to their distinct natures, and in analyzing the model we noticed that we needed to customize the features and consider different theoretical behaviors since we were assessing theoretical thinking in Calculus; the operationalization of the model is discussed below. Clearly, this is not an exhaustive operationalization of the model since the listed theoretical behaviors are pertinent to the questions chosen for the quizzes of this particular study.

Methodology

The study was conducted in an integral Calculus class stretched over one term (thirteen weeks) with an average of thirty five students attending the two classes (1h15 each) per week. The instrument used for the study was a set of quizzes each consisting of a question related to material previously covered in the course. Once a week, students were given a quiz and fifteen minutes of the class time to respond to the question. The time allocated to these quizzes was done in a way so that the course outline was completed as required. It was explained that taking the quiz was optional and that students would be rewarded with 'bonus marks' on their course grade for a complete response or an incomplete response containing valid arguments. The quiz questions were of a conceptual nature, aimed at prompting a type of behavior that we characterize as a display of theoretical thinking. There was usually not a single solution path that had to be followed, but students were asked to "justify" their answers and generally be as expressive as possible. We show three of the quiz questions in the table below.

Is it true that $\int_a^b f(x)dx + \int_c^d g(x)dx = \int_c^d g(x)dx + \int_a^b f(x)dx$ where a, b, c, and d are real numbers?	Explain, in your own words, why this theorem is true: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$	If $g(x)$ is continuous for all real numbers and $\int_1^{\infty} g(x)dx$ is convergent, is $\int_{10}^{\infty} g(x)dx$ is also convergent?

Table 2 – Examples of quiz questions (from left to right: questions 2, 7, and 11)

Once corrected by the instructor, quizzes were returned to the students with a grade and written suggestions for improving the quality of their responses when they were inadequate. Students were not provided with an answer to the questions as we believed that this could inhibit their own creativity and possibly encourage them to mimic the instructor's answers or style.

Responses were analyzed based on our model: In the operationalization of the model we identified a total of thirteen theoretical behaviors (TB). To justify that a TB was indicated by a response, we created an *a priori* list of phrases ('features of discourse') describing possible elements of discourse in the response to each question; we considered these indicative of theoretical thinking and interpreted the performance of these actions as a display of a particular type of TB. Features of discourse that display the same TB were grouped together (as shown in Table 3). Due to space constraints, we display our model (Table 4) with only samples of TBs that correspond to the features of theoretical thinking.

Feature of TT	TB	**Features of discourse** (of possible answers to different questions)
TT1 Reflective	$TB1_2$ Generalizing a solution	• Writing a single *general* statement for positive and negative series by considering the absolute value of terms • Indicating that integral[a,∞) is convergent for *all* $a \geq 1$ • Indicating that the integral of g is convergent over *any* subinterval of [1, ∞) • Remarking that the addition of *any* non-zero constant to the integrand would result in a diverging integral

Table 3 – Sample of how our model is operationalized with features of discourse and TBs.

Feature of TT	**Samples of corresponding TB**
TT1 Reflective	$TB1_2$ Generalizing a solution
TT2 Systemic	
• TT21 Definitional	$TB21_1$ Referring to definitions when deciding upon meaning
• TT22 Proving	$TB22_1$ Engaging in a proving or reasoning activity
• TT23 Hypothetical	$TB23_1$ Being aware of the conditional character of a mathematical statement
• TT24 Contextual	$TB24_1$ Modeling a problem
TT3 Analytic	$TB3_2$ Being sensitive to logical connectives, particularly to implication and its negation

Table 4 – Our model of theoretical thinking, including sample theoretical behaviors

Results and analysis

Two types of analysis were carried out so far; a *question* analysis determining which TBs (and thus types of TT) each question invited, and a *class* analysis: whether (and how) the class engaged in theoretical thinking in responding to each question. At a later stage an analysis of each student's engagement in theoretical thinking across the quizzes will be carried out.

Table 5 indicates how many times each TB was invited by each question (if at all) and overall, as well as how many times each category of theoretical thinking was invited. These are indicated by "count Q*X*", "count TB", and "count TT" respectively (due to space constraints, details for questions 2 and 7 only are shown). For instance, our analysis showed that responding to question 7 could involve representing a situation graphically. This was identified with $TB24_1$ "Modeling a problem" which corresponds to "Contextual thinking"; a feature of Systemic thinking. $TB24_1$ thus appears once in question 7:

	REFLECTIVE			SYSTEMIC								ANALYTIC	
	$TB1_1$	$TB1_2$	$TB1_3$	$TB21_1$	$TB22_1$	$TB22_2$	$TB23_1$	$TB23_2$	$TB24_1$	$TB24_2$	$TB24_3$	$TB3_1$	$TB3_2$
Count Q2	1		2				1			3			
Count Q7	1			1	1	1			1	1			1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Count TB	12	4	4	4	9	2	3	3	9	11	3	1	1
Count TT	20			44								2	

Table 5 – Number of times TBs (and TT) are invited in questions 2 and 7, and by questions overall

Tables 6a and 6b indicate the class's engagement in TT in questions 2 and 7 respectively; where "class TB count" indicates the number of students who displayed the corresponding TB, and "count TT" the total number of times the corresponding type of theoretical thinking was displayed in student responses.

Student count = 42	REFLECTIVE			SYSTEMIC				ANALYTIC
TB invited by question	$TB1_{11}$	$TB1_{3(a)}$	$TB1_{3(b)}$	$TB23_1$	$TB24_{2(a)}$	$TB24_{2(b)}$	$TB24_{2(c)}$	n/a
Class TB count	3	0	11	7	0	21	1	
Count TT	14			29				

Table 6a – Number of students displaying TB, number of times TT was displayed – Question 2

Student count = 31	REFLECTIVE	SYSTEMIC					ANALYTIC
TB invited by question	$TB1_{13}$	$TB21_1$	$TB22_1$	$TB22_2$	$TB24_1$	$TB24_2$	$TB3_2$
Class TB count	8	8	4	3	1	9	0
Count TT	8	25					0

Table 6b – Number of students displaying TB, number of times TT was displayed – Question 7

These preliminary results indicate that the questions succeeded in engaging students in theoretical thinking with a highest occurrence of *systemic* thinking (which was indeed the type of thinking that was most invited by the questions). Table 6a indicates that at least 21 out of 42 students were engaged in (systemic) thinking in question 2. Likewise, Table 6b indicates that at least 9 out of 27 students were engaged in (systemic) thinking in question 7. The results also show that some TBs were more popular than others, and that in general some questions were more effective at provoking theoretical thinking in students than others. This is perhaps an indication that particular features of a question make it more (or less) engaging, which could be a call for further investigation.

Implications for teaching

College-level Calculus instructors are often compelled to certain teaching constraints- we do not deny this; rather we take on a different perspective: We have shown that this reality need not stand in the way of incorporating what we believe to be an essential part of a Calculus course; a theoretical thinking component. Our study shows that by simply posing additional non-routine tasks chosen in a way to promote theoretical thinking, creating a space in which students can actively engage in theoretical thinking (should they wish to do so) is indeed possible, despite these constraints.

Questions for the audience

1- What could be a different type of analysis, yielding additional (or different) results?
2- How can we effectively measure a student's *progress* in engaging in theoretical thinking?
3- Could a different choice of a model of TT change/ enrich the results of this study? If so, how?

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