Understanding Mathematical Conjecturing

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In this study, we open up discussions regarding one of the unexplored aspects of mathematical sophistication, the inductive work of conjecturing. We consider the following questions: What does conjecturing entail? How do the conjectures of experts and novices differ? What characteristics, behaviors, practices, and viewpoints distinguish novice from expert conjecturers? and What activities enable individuals to make conjectures? To answer these questions, we conducted a qualitative research study of eight participants at various levels of mathematical maturity. Answers to our research questions will begin to provide an understanding about what helps students develop the ability to make mathematical conjectures and what characteristics of tasks and topics may effectively elicit such behaviors, informing curriculum development, assessment, and instruction.

Key words: Mathematical sophistication, Enculturation, Mathematical behaviors

Background

Much of contemporary research is concerned with helping students become more adept at problem solving and learning mathematical ideas. In fact, an ongoing concern is empowering students to develop deep understanding of mathematical concepts, instead of only developing shallow procedural proficiency; that is, we want students to be able to apply their knowledge to solve new problems.

From a sociocultural perspective (Bauersfeld, 1995), we argue that this challenge can be in part met by understanding the practices of the mathematical community. As we identify those practices that enable mathematicians to *do* mathematics and find ways to instill these in our students, we will empower them mathematically.

Many researchers have argued this point. Citing Cobb, Bowers, Lave, and Wenger, Rasmussen et al. (2005) argued that learning mathematics is synonymous with participation in mathematical practices; in other words, many of the activities used by the mathematical profession to build new mathematical artifacts are needed by learners to acquire those same artifacts. Carlson and Bloom (2005) argued that successful problem solving involves more than content knowledge; it requires cognitive control skills, methods, and heuristics. It is these mathematically sophisticated behaviors (such as conjecturing, testing, and modeling) that empower problem solvers to correct their own models and arrive at solutions (Moore et al., 2009).

Researchers have also observed and evidenced this. Seaman and Szydlik (2007) noted that even given ample time and resources, preservice teachers failed to relearn forgotten, common, elementary school mathematics concepts and skills because they lacked *Mathematical Sophistication*, that is habits of mind and practices of the mathematics community that *would have* empowered them to acquire mathematical knowledge; these practices include: making sense of definitions, seeking to understand patterns and structure, making analogies, making and testing conjectures, creating mental and physical models (examples and nonexamples of things), and seeking to understand why relationships make sense. Schoenfeld (1992) noted something similar, namely that novices lacked the skills and behaviors characteristic of expert mathematicians, skills which go beyond simple content knowledge, such as: attending carefully to language, building models and examples, making and testing conjectures, and making arguments based on the structure of a problem. Thus, mathematical sophistication is not only critical for prospective mathematicians, but for *anyone* who must engage in mathematical learning and problem solving.

In a recent study, Szydlik, Kuennen, Belnap, Parrott, and Seaman (2012) developed a measure of basic levels of mathematical sophistication and in doing so, found evidence that mathematical sophistication can be developed during the course of a class. So, in order to empower students to more effectively learn mathematics, we must understand the practices of the mathematical community and find ways for them to acquire these practices.

The body of mathematical knowledge develops as professional mathematicians engage in a variety of activities, most of which could be classified as either inductive or deductive work. The inductive work of mathematics involves activities that generate new mathematical ideas; through investigation, exploration, or study, mathematicians create conjectures (i.e. new and unproven hypotheses). Through deductive work, mathematicians take assumed mathematical ideas (axioms) and proven mathematical facts (theorems) and either prove or disprove those conjectures. Proven theorems then become new artifacts in the mathematical knowledge base.

In this study, we have chosen to focus our efforts on the mathematical activity of conjecturing for three main reasons. First, conjecturing is critical to the field of mathematics; it represents the potential mathematical knowledge of the field. Second, conjecturing tends to be a neglected aspect of mathematics classes and enculturation. Most lower-level mathematics courses focus on understanding and applying known theorems. At the upper-level, most attention is devoted to proofs and logic because of their complex and problematic nature, thus focusing predominantly on understanding content, testing conjectures, developing logical arguments, developing counterexamples, and writing deductive arguments--a focus mirrored by the research literature (Weber, 2001; Selden & Selden, 2003; Alcock & Weber, 2005). Third, (in theory) conjecturing is more accessible to undergraduates and novices because it is the generation of ideas which need not be certain; it is hypothesis-making and does not require all the intricacies that accompany deductive work.

This paper focuses on answering the following question: What does mathematical sophistication look like for conjecturing? To answer this question, we consider the following questions: What does conjecturing entail? How do the conjectures of experts and novices differ?

What characteristics, behaviors, practices, and viewpoints distinguish novice from expert conjecturers? and What activities enable individuals to make conjectures?

Methodology

To explore conjecturing, we conducted a qualitative research study, during winter semester 2012, in a mathematics department at a public university. We purposively selected eight participants at various levels of mathematical maturity: two undergraduate students (novice mathematicians), Scott and Laura; three graduate students (apprentice mathematicians), Ann, Noah, and Charlie; and three research mathematicians (i.e. expert mathematicians) Sam, Josh, and Lisa. Student participants were selected from a list of volunteers, who were enrolled in one of two specific courses; undergraduate students were taking a 200-level introduction to proofs course, while graduate students were enrolled in a graduate course in advanced Euclidean geometry, which incorporated a conjecturing component.

Participants were purposively selected to represent diversity in gender, background in Euclidean geometry, and area of expertise. Student participants were selected for diversity in their ability (high/medium/low) to do authentic mathematical work, as judged by their instructors and (in the case of graduate students) conjectures produced during conjecturing tasks. Mathematicians were selected to represent diverse areas of mathematical expertise, which included: probability theory, graph theory, and dynamical systems.

Data Collection

Data collection revolved around each participant's involvement in one individual conjecturing task in the area of Euclidean geometry. Participants were presented with a hardcopy of the task and given ample time and resources to work on it, then (after a break) participated in an interview regarding their experience, approach, and conjectures.

The task was intended to create a context where participants could do inductive work leading to the generation of conjectures. The task provided mathematical definitions for three new types of quadrilaterals; one faculty participant had previously encountered one of these quadrilaterals, otherwise all three definitions were novel for each participant. Participants were allowed to explore these definitions, with the goal of writing as many conjectures about them as possible; they were given as much time as they desired, with the exception of Josh, whom we cut off after about two hours.

During the task, participants had open access to the following resources: a ruler and compass; various colored and regular writing instruments; paper; a list of Euclidean postulates, common notions, definitions, and propositions; a glossary of common geometry terms; and GeoGebra, a free dynamic software program for constructing, measuring, and manipulating dynamic geometric objects. Researchers were also available throughout to answer questions on the task and definitions or to help with software usage.

After the task, each participant took a break, during which we prepared for the interview by discussing our observations and the participant's conjectures and making adjustments to the interview questions. The subsequent interview focused on: a) clarifying any behaviors and thinking that were not discernible by outward observation, b) understanding the participant's experience and perspective, c) understanding the written conjectures, and d) gaining details about the conjecturing process.

Data consisted of video recordings, written work, and observation notes. We took three video recordings during each task and subsequent interview: a) a video taken from the side, of them working on the task; b) a top-down recording of their written work; and c) an internal recording of their computer work. A synchronized compilation of these three video recordings served as the primary data source. Secondary data sources included each participant's written work and conjectures along with our observation notes taken during each task and interview.

Data Analysis

We analyzed the data using grounded theory techniques (Strauss & Corbin, 1998). Beginning with Scott (low-level novice), we independently reviewed the coordinated video feeds and made time-stamped annotations describing his behaviors throughout the task. Then we compared our annotations and negotiated differences, using the video and secondary sources of data to triangulate our observations. We began clustering our annotations around common themes to form initial categories of behaviors. We utilized our results to inform our next interview and observations (of Laura).

We repeated this same process working upward (by experience and level) through our participants, by next meeting with Laura (medium-level novice) and then Ann (low-level apprentice). After meeting with each participant, we coded independently by writing time-stamped annotations and applying our emerging categories; doing so, we modified our categories as appropriate to accurately describe the data; each time we made changes, we back-coded prior participants to examine how the framework reflected the data.

By our meeting with Noah (medium-apprentice), we noticed some broad, over-arching themes, so we independently synthesized our analyses of each individual into a vignette or synopsis, describing for each case (participant) the characteristic of each theme. We continued this process, independently writing, then collaboratively negotiating a final synopsis for each participant. As we compiled these synopses, we did a cross-case analysis, examining differences and similarities across the different levels of expertise. We herein present some of these differences as dimensions of the mathematical activity *conjecturing*.

Results

The distinction between novice, apprentice, and expert conjecturer is not a clear linear one, but varies by each of the five dimensions listed in table 1. These include: a) *overall process*, the process and problem-solving approach used during the task; b) *objects created for* *investigation*, the characteristics of and individual's view of objects created during the investigation; c) *nature of observations*, describing the things they noticed, paid attention to, or looked-for; d) *qualities of written conjectures*, mathematical and verbal qualities of a conjecture; and e) *qualifications of written conjectures*, the threshold of conviction required to consider an idea worthy to be considered a conjecture.

The behaviors listed for each dimension in table 1 represent the extremes observed. Each of a dimension's behaviors and characteristics range from unsophisticated to sophisticated by how they did or did not empower the individual to make conjectures. Other behaviors were noted that could be considered along a continuum, representing different degrees of sophistication. Furthermore, individuals exhibited combinations of these characteristics.

Discussion

Learning to conjecture appears to entail a variety of skills, knowledge, and values. It is affected by logic, content knowledge, practices of observation, experience with mathematical language, and persistence. Indeed, conjecturing appears to rely or draw upon many of the characteristics Seaman and Szydlik (2007) and Schoenfeld (1992) identified. Because of this

Table 1: Sophisticated and Unsophisticated Characteristics and Behaviors by Conjecturing Dimension

Dimension	Unsophisticated	Sophisticated
Overall process	Clear linear approach: make sense of task, explore, write conjectures. Random manipulation to stumble upon something.	Complex non-linear approach. Incorporation of other mathematical knowledge. Systematic analysis to discover or scrutinize. Scaling the problem in or out.
Objects created for investigation	Single *static* "prototype" assumed representative. Errors in constructions.	Multiple examples considered and required. Dynamic view of situation and examples.
Nature of observations	Reliance upon appearance *"looks like..."* Naive acceptance of definitions. Focus on superficial properties (e.g. orientation, location, etc.).	Consideration of degenerate (undefined) cases. Measuring or observing precise criteria and properties. Consideration of causal or covariational relationships.
Qualities of written conjectures	Perspective-dependent statements that do not present clear, testable, mathematical criteria.	Precise, testable statements. Conventional use of math terminology.

	Unconventional use of math terms. Reliance on common vernacular.	Inclusion of both general and special cases.
Qualification of conjectures	If you think it or see it once, it is a conjecture.	Level of conviction/scrutiny to be considered a conjecture. Must be an interesting/non-trivial result. Evidenced by multiple cases or a specially-designed dynamic construction.

and its accessibility, it may serve as a type of mathematical activity in which some of these characteristics can be discussed and developed. For example, the articulation of conjectures provides an opportunity to discuss logic and mathematical language; thus perhaps conjecturing could provide an earlier forum to deal with these topics and acclimate students earlier than a transition to proofs course.

One question at the front of our mind is the context. Because of its constructive nature and the many physical and virtual tools available, Euclidean geometry was a natural context for exploration and conjecturing. How could it be incorporated into other areas of mathematics?

In the end, we are left with several unanswered questions: How can conjecturing be fostered and developed? To what extent is conjecturing tied to content area? How can conjecturing be integrated into other branches of mathematics? and How does experience with conjecturing affect learning and success in mathematics?

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